Center of Mass and Spin for Isolated Sources of Gravitational Radiation

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Introduction

In this work we define the notions of center of mass and intrinsic angular momentum for isolated systems, also we obtain their dynamical evolution when gravitational radiation is emitted.

In Newtonian theory and special relativity one can find a particular trajectory with the property that the mass dipole moment vanishes at this trajectory. This special trajectory is called the center of mass. If one would like to generalize this concept to GR, then the goal would be to find a worldline in spacetime with analogous properties to the one described in Newtonian gravity or special relativity.

Thus, to implement this ideas one should generalize the mass dipole moment/angular momentum 2-form to GR, and then define the center of mass worldline as the special place where the mass dipole vanishes. As a bonus one should obtain the intrinsic angular momentum evaluating the non-vanishing part of this generalized 2-form on the center of mass worldline.

The evolution of isolated systems and its gravitational radiation naturally fits with the notion of asymptotically flat spacetimes. Thus, our approach will be based on this mathematical framework.

Asymptotically Flatness and $\mathscr{I}^+$

A spacetime $(\mathscr{M}, g_{ab})$ is called asymptotically flat if the curvature tensor vanishes as infinity is approached along the future-directed null geodesics of the spacetime. These geodesics end up at what is referred to as future null infinity $\mathscr{I}^+$, the future null boundary of the spacetime .

A future null asymptote is a manifold $\hat{\mathscr{M}}$ with boundary $\mathscr{I}^+ \equiv \partial\hat{\mathscr{M}}$ together with a smooth lorentzian metric $\hat{g}_{ab}$, and a smooth function Ω on $\hat{\mathscr{M}}$ satisfying the following

- $\hat{\mathscr{M}} = \mathscr{M} \cup \mathscr{I}^+$
- On $\mathscr{M}$, $\hat{g}_{ab} = \Omega^2 g_{ab}$ with $\Omega > 0$
- At $\mathscr{I}^+$, $\Omega = 0$, $n^*_a \equiv \partial_a\Omega \neq 0$ and $\hat{g}^{ab} n^*_a n^*_b = 0$

Since $\mathscr{I}^+$ is a null hypersurface in the rescaled manifold $\hat{\mathscr{M}}$ the restriction of the rescaled metric on this null boundary takes the form

$$d\hat{s}^2 = \frac{4 d\zeta d\bar{\zeta}}{P^2}. \tag{1}$$

One can introduce a set of coordinates in the neighborhood of $\mathscr{I}^+$. Associated with the NU coordinates $(u, r, \zeta, \bar{\zeta})$, there is a null tetrad system denoted by $(l^*_a, n^*_a, m^*_a, \bar{m}^*_a)$, the first tetrad vector l^*_a is defined as

$$l^*_a = \nabla_a u, \tag{2}$$

Thus, l^{a*} is a null vector tangent to the geodesic of the null surface. The second tetrad vector n^{*a} is normalized to l^{*a}

$$n^*_a l^{*a} = 1. \tag{3}$$

The null tetrad is finally completed with the choice of a complex vector m^{a*} orthogonal to l^{a*} and n^{a*}

$$m^*_a \bar{m}^{*a} = -1, \tag{4}$$

and zero for any other product. The spacetime metric is then given by

$$g_{ab} = l^*_a n^*_b + n^*_a l^*_b - m^*_a \bar{m}^*_b - \bar{m}^*_a m^*_b. \tag{5}$$

A quantity η that transforms as $\eta \to e^{is\lambda}\eta$ under a rotation $m^{a*} \to e^{i\lambda}m^{a*}$ is said to have a spin weight s. For any function $f(u, \zeta, \bar{\zeta})$, we define the differential operators $\eth^*$ and $\bar{\eth}^*$ by

$$\eth^* f = P^{1-s}\frac{\partial(P^s f)}{\partial\zeta}, \tag{6}$$

$$\bar{\eth}^* f = P^{1+s}\frac{\partial(P^{-s} f)}{\partial\bar{\zeta}}, \tag{7}$$

where f has a spin weight s and P is the conformal factor defining the metric. Now, consider a different choice of the original u cut of $\mathscr{I}^+$.

$$u_B = Z(u, \zeta, \bar{\zeta}) \qquad u = T(u_B, \zeta, \bar{\zeta}). \tag{8}$$

where Z is a smooth function and T is the inverse of Z, and where this function satisfaces the relation $\dot{T}Z' = 1$.

Now it is possible to define a new set of null vectors $(l_a, n_a, m_a, \bar{m}_a)$ associated with the coordinates $(u_B, r_B, \zeta, \bar{\zeta})$ where the new null surfaces intersect $\mathscr{I}$ on the $u_B = const$ cuts. These coordinates are the Bondi coordinates, where $u_B = Z(u, \zeta, \bar{\zeta})$ and $r_B = Z' r$. Since the NU and Bondi null tetrad are two different vector basis, we can express any one in term of the other.

$$l_a^* = \frac{1}{Z'}[l_a - \frac{L}{r_B}\bar{m}_a - \frac{\bar{L}}{r_B}m_a + \frac{L\bar{L}}{r_B^2}n_a], \tag{9}$$

$$n_a^* = Z' n_a, \tag{10}$$

$$m_a^* = m_a - \frac{L}{r_B}n_a, \tag{11}$$

$$\bar{m}_a^* = \bar{m}_a - \frac{\bar{L}}{r_B}n_a, \tag{12}$$

where

$$L(u_B, \zeta, \bar{\zeta}) = -\frac{\eth_{(u_B)} T}{\dot{T}} = \eth_{(u)} Z(u, \zeta, \bar{\zeta})|_{u=T(u_B,\zeta,\bar{\zeta})}. \tag{13}$$

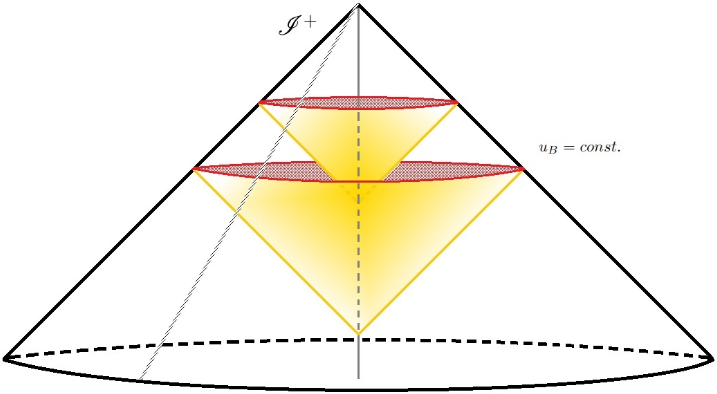

Figura : Bondi family cuts. this family intersects $\mathscr{I}^+$ at $u_B = cte.$

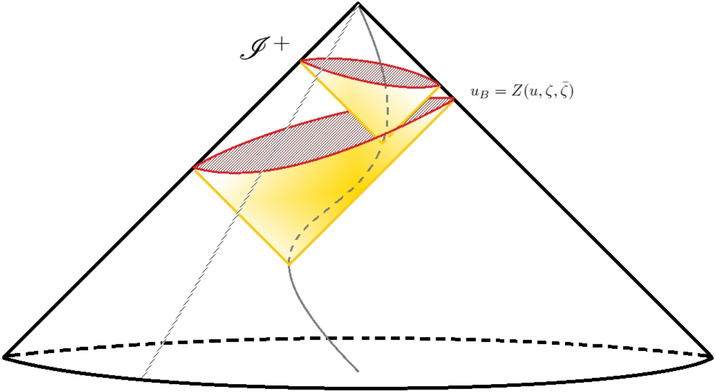

Figura : NU family described from a Bondi system.

The Spin Coefficient Formalism

As the spacetime is assumed to be empty in a neighborhood of $\mathscr{I}^+$ the gravitational field is given by the Weyl tensor. Using the available tetrad one defines five complex scalars, whose asymptotic behavior is

$$\begin{aligned}\psi_0 &= C_{abc}{}^{d} m^a l^b l^c m_d \simeq \frac{\psi_0^0}{r_B^5}, \quad \psi_3 = C_{abc}{}^{d} l^a n^b n^c \bar{m}_d \simeq \frac{\psi_3^0}{r_B^2}.\\ \psi_1 &= C_{abc}{}^{d} n^a l^b l^c m_d \simeq \frac{\psi_1^0}{r_B^4}, \quad \psi_4 = C_{abc}{}^{d} \bar{m}^a n^b n^c \bar{m}_d \simeq \frac{\psi_4^0}{r_B}.\end{aligned}$$

$$\psi_2 = \frac{1}{2}(C_{abc}{}^{d} l^a n^b m^c \bar{m}_d - C_{abcd} l^a n^b l^c n_d) \simeq \frac{\psi_2^0}{r_B^3}.$$

Some of those equations relate the Weyl scalars with the Bondi shear, i.e.

$$\psi_2^0 + \eth^2 \bar{\sigma}^0 + \sigma^0 \dot{\bar{\sigma}}^0 = \bar{\psi}_2^0 + \bar{\eth}^2 \sigma^0 + \bar{\sigma}^0 \dot{\sigma}^0, \tag{14}$$

$$\psi_3^0 = \eth \dot{\bar{\sigma}}^0, \tag{15}$$

$$\psi_4^0 = -\ddot{\bar{\sigma}}^0, \tag{16}$$

with σ^0 the value of the Bondi shear at null infinity.

$$\sigma = m^a m^b \nabla_a l_b \simeq \frac{\sigma^0}{r_B^2} \tag{17}$$

This complex scalar is called the Bondi free data since $\ddot{\sigma}^0$ yields the gravitational radiation reaching null infinity.

In the same way we can define the Weyl scalars in NU using the fact that the Weyl tensor ${C_{abc}}^d$ is conformally invariant.

$$\begin{aligned}\psi_1^* &= {C_{abc}}^d n^{a*} l^{b*} l^{c*} m_d^* \simeq \psi_1^{0*} r^{-4},\\ \sigma^* &= m^{*a} m^{*b} \nabla_a l_b^* \simeq \sigma^{0*} r^{-2}.\end{aligned}$$

From the tetrad equations, we can find transformations from NU to Bondi for any scalar or spin coefficient. In particular we are interested in

$$\frac{\psi_1^{0*}}{Z'^3} = [\psi_1^0 - 3L\psi_2^0 + 3L^2\psi_3^0 - L^3\psi_4^0], \tag{18}$$

where ψ_1^{0*} is constructed from the N-U tetrad. Similarly we find the relation between σ^{0*} and σ^0

$$\frac{\sigma^{*0}}{Z'} = \sigma^0 - \eth^2 Z. \tag{19}$$

where σ^{0*} is the NU shear.

Finally, the Bianchi identities (in Bondi coordinates) are given by

$$\dot{\psi}_0^0 = -\eth\psi_1^0 + 3\sigma^0\psi_2^0, \tag{20}$$
$$\dot{\psi}_1^0 = -\eth\psi_2^0 + 2\sigma^0\psi_3^0, \tag{21}$$
$$\dot{\psi}_2^0 = -\eth\psi_3^0 + \sigma^0\psi_4^0. \tag{22}$$

Note that eq. (14) defines a real variable Ψ called the mass aspect.

$$\Psi = \psi_2^0 + \eth^2\bar{\sigma}^0 + \sigma^0\dot{\bar{\sigma}}^0, \tag{23}$$

In term of Ψ is possible to write the Bondi Mass M and Bondi lineal momentum P^i by

$$M = -\frac{c^2}{8\pi\sqrt{2}G}\int \Psi\, dS, \tag{24}$$
$$P^i = -\frac{c^3}{8\pi\sqrt{2}G}\int \Psi\tilde{l}^i\, dS, \tag{25}$$

Another important variable in our construction is the null cone cut of null infinity. The leading contribution to the solution comes from the Huygens part of the Null Cone Cuts equation

$$\bar{\eth}^2\eth^2 Z = \bar{\eth}^2\sigma^0(Z,\zeta,\bar{\zeta}) + \eth^2\bar{\sigma}^0(Z,\zeta,\bar{\zeta}), \tag{26}$$

In particular, the kernel of the RNC cuts is a 4 dim space x^a, i.e. a flat cut

$$Z_0 = x^a\ell_a, \qquad x^a = (R^0, R^i), \qquad \ell_a = (Y^0_0, -\frac{1}{2}Y^0_{1i}).$$

Thus, if $x^a(u)$ describes a worldline in the solution space, the function $Z(x^a(u),\zeta,\bar{\zeta})$ describes a one parameter family of NU cuts. Solutions of this equation are very difficult to obtain in closed form. We thus search for perturbative solutions

$$Z = Z_0 + Z_1 + Z_2 + ..., \tag{27}$$

where each term in the series is determined from the previous one and the free data $\sigma^0(u_B,\zeta,\bar{\zeta})$. The first two terms satisfy

$$\bar{\eth}^2\eth^2 Z_0 = 0, \tag{28}$$

$$\bar{\eth}^2\eth^2 Z_1 = \bar{\eth}^2\sigma^0(Z_0,\zeta,\bar{\zeta}) + \eth^2\bar{\sigma}^0(Z_0,\zeta,\bar{\zeta}). \tag{29}$$

The first perturbative order Z_1 is given by,

$$Z_1 = R^0 - \frac{1}{2}R^i Y^0_{1i} + \left(\frac{\sigma_R^{ij}}{12} + \frac{\sqrt{2}}{72}\dot{\sigma}_I^{ig} R^f \epsilon^{gfj}\right) Y^0_{2ij}, \tag{30}$$

Given a $u = const.$ null foliation, which can be either NU or Bondi, introducing an affine parameter r and constructing the $r = const.$ 2-surface with surface element $l^{*[a}n^{*b]}dS$, the linkage integral is defined as

$$L_\xi(\mathscr{I}^+) = -\frac{1}{16\pi}\lim_{r\to\infty}\int\left(\nabla^{[a}\xi^{b]} + \nabla_c\xi^c\hat{l}^{*[a}\hat{n}^{*b]}\right)\hat{l}^*_a\hat{n}^*_b\,dS, \tag{31}$$

with ξ^a the asymptotic Killing vector. This vector satisfies the asymptotic Killing equation

$$\xi_{a;b} + \xi_{b;a} = O(r^{-n}) \tag{32}$$

$$(\xi_{a;b} + \xi_{b;a})l^{b*} = 0. \tag{33}$$

Directly from the linkage integral we define the mass dipole momentum and the angular momentum

$$D^{*i} + ic^{-1}J^{*i} = -\frac{c^2}{12\sqrt{2}G}\left[\frac{2\psi_1^0 - 2\sigma^0\eth\bar{\sigma}^0 - \eth(\sigma^0\bar{\sigma}^0)}{Z'^3}\right]^{*i}. \tag{34}$$

Definitions of Center of Mass and Angular Momentum

Exist a special worldline in the N-U foliation such that, at each $u = const.$ cut, the mass dipole momentum D^{*i} vanishes. This special worldline will be called the center of mass worldline. The angular momentum J^{i*} evaluated at the center of mass will be called intrinsic angular momentum S^i.

The center of mass worldline is then determined from,

$$Re\left[\frac{2\psi_1^0 - 2\sigma^0\eth\bar{\sigma}^0 - \eth(\sigma^0\bar{\sigma}^0)}{Z'^3}\right]^{*i} = 0. \quad (35)$$

Likewise, the intrinsic angular momentum is given by

$$S^i = -\frac{c^3}{12\sqrt{2}G} Im\left[\frac{2\psi_1^0 - 2\sigma^0\eth\bar{\sigma}^0 - \eth(\sigma^0\bar{\sigma}^0)}{Z'^3}\right]^{*i}, \quad (36)$$

evaluated at the center of mass worldline. To write down the mass dipole moment and and angular momentum in Bondi coordinates it is convenient to define analogous quantities in a Bondi tetrad, i.e.,

$$D^i + ic^{-1}J^i = -\frac{c^2}{12\sqrt{2}G}\left[2\psi_1^0 - 2\sigma^0\eth\bar{\sigma}^0 - \eth(\sigma^0\bar{\sigma}^0)\right]^i. \quad (37)$$

Using the relations between the NU and the Bondi null vectors to transform the quantities $(\psi_1^{0*}, \sigma^{0*}, \eth^*) \rightarrow (\psi_1^0, \sigma^0, \eth)$, one can write as

$$D^{*i}(u) = D^i(u_B) + \frac{3c^2}{6\sqrt{2}G} Re[\eth Z(\Psi - \eth^2\bar{\sigma}^0) + F]^i \quad (38)$$

$$J^{i*}(u) = J^i(u_B) + \frac{3c^3}{6\sqrt{2}G} Im[\eth Z(\Psi - \eth^2\bar{\sigma}^0) + F]^i \quad (39)$$

with

$$F = -\frac{1}{2}(\sigma^0\eth\bar{\eth}^2 Z + \eth^2 Z\eth\bar{\sigma}^0 - \eth^2 Z\eth\bar{\eth}^2 Z) - \frac{1}{6}(\bar{\sigma}^0\eth^3 Z + \bar{\eth}^2 Z\eth\sigma^0 - \bar{\eth}^2 Z\eth^3 Z). \quad (40)$$

If we insert the center of mass RNC cut Z_1 in (38), then its l.h.s. vanishes on a $u =$ const. surface and we obtain an algebraic equation to be solved for $R^i(u)$. Equation (39) then gives a relationship between S^i and J^i, the intrinsic and total angular momentum respectively.

Approximations and Assumptions

Although the main equations have been presented above, to obtain the explicit form of the worldline in this work we will make the following assumptions.

- $\sigma = 0$ for some initial Bondi time, usually taken to be $-\infty$.
- R^i is a small deviation form the coordinates origin.
- $R^0 = u$ assuming the slow motion approximation.
- The Bondi shear only has a quadrupole term.

Using the tensorial spin-s spherical harmonics Y^0_0, Y^0_{1i}, Y^0_{2ij}, etc., one can expand the relevant scalars at null infinity as

$$\begin{aligned} \sigma^0 &= \sigma^{ij}(u_B)\, Y^2_{2ij}(\zeta,\bar\zeta), \\ \psi^0_1 &= \psi^{0i}_1(u_B)\, Y^1_{1i}(\zeta,\bar\zeta) + \psi^{0ij}_1(u_B)\, Y^1_{2ij}(\zeta,\bar\zeta), \\ \Psi &= -\frac{2\sqrt{2}G}{c^2}M - \frac{6G}{c^3}P^i\, Y^0_{1i}(\zeta,\bar\zeta) + \Psi^{ij}(u_B)\, Y^0_{2ij}(\zeta,\bar\zeta), \end{aligned} \tag{41}$$

So, we write the linearized solution $Z = u + \delta u$ with

$$\delta u \equiv -\frac{1}{2}R^i(u)\, Y^0_{1i}(\zeta,\bar\zeta) + \frac{1}{12}\sigma^{ij}_R(u)\, Y^0_{2ij}(\zeta,\bar\zeta), \tag{42}$$

Main Results

Then we make a Taylor expansion of the Bondi tetrad variables up to first order in δu, we take the real and the imaginary part of the $l=1$ component of eq. and use that D^{i*} vanishes at a $u = const.$ cut to write

$$
\begin{aligned}
MR^i &= D^i + \frac{8}{5\sqrt{2}c}\sigma_R^{ij}P^j, & (43)\\
J^i &= S^i + R^j P^k \epsilon^{ijk}. & (44)
\end{aligned}
$$

Then, using our definition of center of mass in the Bianchi identities and solving for the real and imaginary $l=1$ component, obtain

$$
\begin{aligned}
\dot{D}^i &= P^i, & (45)\\
\dot{J}^i &= \frac{c^3}{5G}(\sigma_R^{kl}\dot{\sigma}_R^{jl} + \sigma_I^{kl}\dot{\sigma}_I^{jl})\epsilon^{ijk}. & (46)
\end{aligned}
$$

In the same way taking the $l = 0, 1$ part of the $\dot{\Psi}$ yields the mass loss equation and the linear momentum time rate, namely,

$$
\begin{aligned}
\dot{M} &= -\frac{c}{10G}(\dot{\sigma}_R^{ij}\dot{\sigma}_R^{ij} + \dot{\sigma}_I^{ij}\dot{\sigma}_I^{ij}), & (47)\\
\dot{P}^i &= \frac{2c^2}{15G}\dot{\sigma}_R^{jl}\dot{\sigma}_I^{kl}\epsilon^{ijk}. & (48)
\end{aligned}
$$

Now, taking a time derivative of eq. (43), using eq. (45), and writing up to quadratic terms in σ^{ij}, gives

$$M\dot{R}^i = P^i + \frac{8}{5\sqrt{2}c}\dot{\sigma}_R^{ij}P^j, \tag{49}$$

the relationship between the velocity of the center of mass $\dot{R}^i$ and the Bondi momentum. It departs from the newtonian formula by radiation terms.
Finally, taking one more Bondi time derivative of (49) yields the equation of motion for the center of mass,

$$M\ddot{R}^i = \frac{2c^2}{15G}\dot{\sigma}_R^{jl}\dot{\sigma}_I^{kl}\epsilon^{ijk} + \frac{8}{5\sqrt{2}c}\ddot{\sigma}_R^{ij}P^j. \tag{50}$$

The r.h.s. of the equation only depends on the gravitacional data at null infinity and the initial mass of the system.
Similarly, taking a time derivative of (44) together with (46) gives

$$\dot{S}^i = \dot{J}^i = \frac{c^3}{5G}(\sigma_R^{kl}\dot{\sigma}_R^{jl} + \sigma_I^{kl}\dot{\sigma}_I^{jl})\epsilon^{ijk}. \tag{51}$$

In the PN equations the radiative energy loss, the linear and angular momentum loss are given by (in units of $G = c = 1$)

$$\dot{E}_{PN} = -\frac{1}{5}\,\dot{U}^{ij}\,\dot{U}^{ij} - \frac{16}{45}\,\dot{V}^{ij}\,\dot{V}^{ij} - \frac{1}{189}\,\dot{U}^{ijk}\,\dot{U}^{ijk} - \frac{1}{84}\,\dot{V}^{ijk}\,\dot{V}^{ijk} \tag{52}$$

$$\dot{P}^{i}_{PN} = \left(\frac{16}{45}\,\dot{U}^{kl}\,\dot{V}^{jl} + \frac{1}{126}\,\dot{U}^{klm}\,\dot{V}^{jlm}\right)\epsilon^{ijk} - \frac{2}{63}(\,\dot{U}^{jk}\,\dot{U}^{ijk} + 2\,\dot{V}^{jk}\,\dot{V}^{ijk}) \tag{53}$$

$$\dot{J}^{i}_{PN} = -\left(\frac{2}{5}\,U^{kl}\,\dot{U}^{jl} + \frac{32}{45}\,V^{kl}\,\dot{V}^{jl}\right)\epsilon^{ijk} - \left(\frac{1}{63}\,U^{klm}\,\dot{U}^{jlm} + \frac{1}{28}\,V^{klm}\,\dot{V}^{jlm}\right)\epsilon^{ijk} \tag{54}$$

where in the above equations the quadrupole as well as octupole terms have been included.

To compare both approaches, we must include in our formalism the octupole contribution to the equations for the mass, angular and linear momentum

$$\dot{M} = -\frac{1}{10}(\dot{\sigma}_R^{ij}\dot{\sigma}_R^{ij} + \dot{\sigma}_I^{ij}\dot{\sigma}_I^{ij}) - \frac{3}{7}(\dot{\sigma}_R^{ijk}\dot{\sigma}_R^{ijk} + \dot{\sigma}_I^{ijk}\dot{\sigma}_I^{ijk}), \quad (55)$$

$$\dot{P}^i = -\frac{2}{15}\dot{\sigma}_R^{kl}\dot{\sigma}_I^{jl}\epsilon^{ijk} - \frac{\sqrt{2}}{7}(\dot{\sigma}_R^{jk}\dot{\sigma}_R^{ijk} + \dot{\sigma}_I^{jk}\dot{\sigma}_I^{ijk}) - \frac{3}{7}\dot{\sigma}_R^{klm}\dot{\sigma}_I^{jlm}\epsilon^{ijk}. \quad (56)$$

$$\dot{J}^i = \frac{1}{5}(\sigma_R^{kl}\dot{\sigma}_R^{jl} + \sigma_I^{kl}\dot{\sigma}_I^{jl})\epsilon^{ijk} + \frac{9}{7}(\sigma_R^{klm}\dot{\sigma}_R^{jlm} + \sigma_I^{klm}\dot{\sigma}_I^{jlm})\epsilon^{ijk}. \quad (57)$$

Making the following association

$$\sigma_R^{ij} = -\sqrt{2}U^{ij} \qquad \sigma_I^{ij} = \frac{8}{3\sqrt{2}}V^{ij}$$

$$\sigma_R^{ijk} = -\frac{1}{9}U^{ijk} \qquad \sigma_I^{ijk} = \frac{1}{6}V^{ijk}$$

Final Comments

- We have defined the notion of center of mass and spin for asymptotically flat spacetimes.

- The main tools used in our approach are the linkages together with a canonical NU foliation constructed from solutions to the Regularized Null Cone cut equation. The RNC cut foliation is given in the so called Newman Penrose gauge with a vanishing shear in the asymptotic past.

- We have obtained equations of motion for these variables linking their time evolution to the emitted gravitational radiation.

- We have compared our equations with those derived from the PN formalism, the results are very encouraging since, the r.h.s. of the evolution equation for these variables are identical in both formulations. However, the relationship between total linear momentum and the velocity of the center of mass is different in both approaches.

Thank you