Symmetries in General Relativity

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2. Participants

2.1. ICRANet participants

- Bini Donato (ICRANet and Istituto per le Applicazioni del Calcolo “M. Picone,” CNR, Italy)
- Cherubini Christian (ICRANet and University Campus Biomedico, Rome, Italy)
- Geralico Andrea (ICRANet and University of Rome “La Sapienza,” Italy)
- Jantzen T. Robert (ICRANet and Villanova University, USA)
- Kerr P. Roy (ICRANet)
- Wen-Biao Han (ICRANet)
- Ruffini J. Remo (ICRANet and University of Rome “La Sapienza,” Italy)

2.2. Past collaborators

- Carini Paolo (high school science teacher in San Francisco)
- Cruciani Gianluca (ICRANet and high school teacher in Perugia, Italy)
- Ehlers Jurgen (Max Planck Institut für Gravitationsphysik, Golm, Germany, † 2008)
- Merloni Andrea (post-doc at the Max Planck Institute for Extraterrestrial Physics, Germany)
- Miniutti Giovanni (researcher at the Institute of Astronomy, Cambridge UK)
- Spoliti Giuseppe (commercial director at RHEA Systems, S.A.)

2.3. Ongoing collaborations

- De Felice Fernando (University of Padova, Italy)
- Fortini Pierluigi (University of Ferrara, Italy)
- Lusanna Luca (INFN, Florence, Italy)
- Ruggiero M. Luca (Politecnico of Torino, Italy)
- Ortolan Antonello (INFN, Padova, Italy)
- Stella Luigi (Astronomic Observatory of Rome, Italy)
- Tartaglia Angelo (Politecnico of Torino, Italy)
- Esposito Giampiero (INFN, Napoli, Italy)
- Damour T. (IHES, Paris)

2.4. Students

- Gregoris Daniele (Irap PhD student)
- Boshkayev Kuantay (Irap PhD student)
- Haney Maria (Irap PhD student)

3. Brief description

3.1. Spacetime splitting techniques in General Relativity

Spacetime splitting techniques play a central role and have fundamental interest in general relativity in view of extracting from the unified notion of spacetime the separate classical notions of space and time, at the foundation of all of our experience and intuition. Studying all the existing different approaches scattered in the literature has allowed the creation a unique framework encompassing all of them [1] and a more clear geometrical interpretation of the underlying "measurement process" for tensors and tensorial equations. "Gravitoelectromagnetism" is a convenient name for this framework because it helps explain the close relation between gravity and electromagnetism represented by the Coriolis and centrifugal forces on one side and the Lorentz force on the other side.

3.1.1. "1+3" splitting of the spacetime

During the last century, the various relativistic schools: Zelmanov, Landau, Lifshitz and the Russian school, Lichnerowicz in France, the British school, the Italian school (Cattaneo and Ferrarese), scattered Europeans (Ehlers and Trautman, for example) and the Americans (Wheeler, Misner, etc.), developed a number of different independent approaches to spacetime splitting almost without reference to each other.

R. Ruffini [2], a former student of Cattaneo and a collaborator of Wheeler, looking for a better understanding of black holes and their electromagnetic properties, stimulated Jantzen, Carini and Bini to approach the problem and to make an effort to clarify the interrelationships between these various approaches as well as to shed some light on the then confusing works of Abramowicz and others on relativistic centrifugal and Coriolis forces. By putting them all in a common framework and clarifying the related geometrical aspects, some order was brought to the field [1, 3, 4].

3.1.2. Measurement process in general relativity

The investigations on the underlying geometrical structure of any spacetime splitting approach show that it is not relevant to ask which of these various splitting formalisms is the "best" or "correct" one, but to instead ask what exactly each one of them "measures" and which is specially suited to a particular application.

For instance, in certain situations a given approach can be more suitable than another to provide intuition about or simplify the presentation of the invariant spacetime geometry, even if all of them may always be used. These ideas were then used to try to understand better the geometry of circular orbits in stationary spacetimes and their physical properties where the connection between general relativity and its Newtonian progenitor are most natural.

The list of problems approached and results obtained together can be found in Appendix A.

3.2. Motion of particles and extended bodies in general relativity

The features of test particle motion along a given orbit strongly depend on the nature of the background spacetime as well as on the model adopted for the description of the intrinsic properties of the particle itself (e.g., its charge or spin). As a basic assumption, the dimensions of the test particle are supposed to be very small compared with the characteristic length of the background field in such a way that the background metric is not modified by the presence of the particle (i.e., the back reaction is neglected), and that the gravitational radiation emitted by the particle in its motion is negligible. The particle can in turn be thought as a small extended body described by its own energy-momentum tensor, whose motion in a given background may be studied by treating the body via a multipole expansion. Thus, a single-pole particle is a test particle without any internal structure; a pole-dipole particle instead is a test particle whose internal structure is expressed by its spin, and so on. The equations of motion are then obtained by applying Einsteins field equations together with conservation of the energy-momentum tensor describing the body. For a single-pole particle this leads to a free particle moving along the geodesics associated with the given background geometry. The motion of a pole-dipole particle is instead described by the Mathisson-Papapetrou-Dixon equations which couple background curvature and the spin tensor of the field. The motion of particles with an additional quadrupolar structure has been developed mostly by Dixon; because of its complexity, there are very few applications in the literature. Finally, the discussion of the case in which the test particle also has charge in addition to spin or mass quadrupole moment is due to Dixon and Souriau and this situation has been very poorly studied as well.

A complete list of the original results obtained and a deeper introduction to the models can be found in Appendix B.

3.2.1. Test particles

Since the 1990s we have been investigating the geometrical as well as physical properties of circular orbits in black hole spacetimes, selecting a number of special orbits for various reasons. These were already reviewed in a previous ICRANet report on activities. A recent work has instead been to consider a given gravitational background a (weak) radiation field superposed on it and a test particle interacting with both fields. Interesting effects arise like the Poynting-Robertson effect which have been considered in the framework of the full general relativistic theory for the first time.

Poynting-Robertson effect can be briefly described as follows.

For a small body orbiting a star, the radiation pressure of the light emitted by the star in addition to the direct effect of the outward radial force exerts a drag force on the body's motion which causes it to fall into the star unless the body is so small that the radiation pressure pushes it away from the star. Called the Poynting-Robertson effect, it was first investigated by J.H. Poynting in 1903 using Newtonian gravity and then later calculated in linearized general relativity by H.P. Robertson in 1937. These calculations were revisited by Wyatt and Whipple in 1950 for applications to meteor orbits, making more explicit Robertson's calculations for slowly evolving elliptical orbits and slightly extending them.

The drag force is easily naively understood as an aberration effect: if the body is in a circular orbit, for example, the radiation pressure is radially outward from the star, but in the rest frame of the body, the radiation appears to be coming from a direction slightly towards its own direction of motion, and hence a backwards component of force is exerted on the body which acts as a drag force. If the drag force dominates the outward radial force, the body falls into the star. For the case in which a body is momentarily at rest, a critical luminosity similar to the Eddington limit for a star exists at which the inward gravitational force balances the outward radiation force, a critical value separating radial infall from radial escape. Similarly for a body initially in a circular orbit, there are two kinds of solutions: those in which the body spirals inward or spirals outward, depending on the strength of the radiation pressure.

We have considered [156, 162, 159] this problem in the context of a test body in orbit in a spherically symmetric Schwarzschild spacetime without the restriction of slow motion, and then in the larger context of an axially symmetric Kerr spacetime while developing the equations for a more general stationary axially symmetric spacetime. The finite size of the radiating body is ignored.

We have also developed applications to cylindrically symmetric Weyl class spacetimes (exhibiting a typical naked singularity structure) as well as to the Vaidya radiating spacetime where the photon field is not a test field but the source of the spacetime itself.

3.2.2. Spinning test particles

During the last five years we have investigated the motion of spinning test particles along special orbits in various spacetimes of astrophysical interest: black hole spacetimes as well as more "exotic" background fields representing naked singularities or the superposition of two or more axially symmetric bodies kept apart in a stable configuration by gravitationally inert singular structures.

In particular, we have focused on the so called "clock effect," defined by the difference in the arrival times between two massive particles (as well as photons) orbiting around a gravitating source in opposite directions after one complete loop with respect to a given observer [5, 6, 7].

We have also analyzed the motion of massless spinning test particles, according to an extended version of the Mathisson-Papapetrou model in a general vacuum algebraically special spacetime using the Newman-Penrose formalism in the special case in which the multipole reduction world line is aligned with a principal null direction of the spacetime. Recent applications concern instead the study of the Poynting-Robertson effect for spinning particles.

3.2.3. Particles with both dipolar and quadrupolar structure (Dixon's model)

We have studied the motion of particles with both dipolar and quadrupolar structure in several different gravitational backgrounds (including Schwarzschild, Kerr, weak and strong gravitational waves, etc.) following Dixon's model and within certain restrictions (constant frame components for the spin and the quadrupole tensor, center of mass moving along a circular orbit, etc.).

We have found a number of interesting situations in which deviations from geodesic motion due to the internal structure of the particle can give rise to measurable effects.

3.2.4. Exact solutions representing extended bodies with quadrupolar structure

We have investigated geometrical as well as physical properties of exact solutions of Einstein's field equations representing extended bodies with structure up to the quadrupole mass moment, generalizing so the familiar black hole spacetimes of Schwarzschild and Kerr.

Recent results involve the use of the equivalence principle to compare geodesic motion in these spacetimes with nongeodesic motion of structured particles in Schwarzschild and Kerr spacetimes, allowing an interesting analysis which strongly support Dixon's model.

3.3. Perturbations

A discussion of curvature perturbations of black holes needs many different approaches and mathematical tools. For example, the Newman-Penrose formalism in the tetradic and spinor version, the Cahen-Debever-Defrise self-dual theory, the properties of the spin-weighted angular harmonics, with particular attention to the related differential geometry and the group theory, some tools of complex analysis, etc. Furthermore, even using any of the above mentioned approaches, this remains a difficult problem to handle. It is not by chance, for instance, that the gravitational and electromagnetic perturbations of the Kerr-Newman rotating and charged black hole still represent an open problem in general relativity.

During the last years, however, modern computers and software have reached an exceptional computational level and one may re-visit some of these still open problems, where technical difficulties stopped the research in the past. Details can be found in Appendix C.

3.3.1. Curvature and metric perturbations in algebraically special spacetimes

Most of the work done when studying perturbations in General Relativity concerns curvature perturbations from one side or metric perturbations from the other side. In the first case, one can easily deal with gauge invariant quantities but the problem of finding frame-independent objects arises. Furthermore, the reconstruction of the metric once the curvature perturbations are known is a very difficult task. In the second case, instead, in order to start working with an explicit metric since the beginning, the choice of a gauge condition is necessary. Gauge independent quantities should therefore be determined properly.

There exist very few examples of works considering both cases of curvature and metric perturbations on the same level so that we have been motivated to start working in this direction.

3.3.2. Curvature perturbations in type D spacetimes

In the Kerr spacetime Teukolsky [8] has given a single "master equation" to deal with curvature perturbations by a field of any spin ("spin-weight," properly speaking). Then the problem of extending the results of Teukolsky to other spacetimes is raised.

Actually, a very important result that we have obtained framed the Teukolsky equation in the form of a linearized de-Rham Laplacian equation for the perturbing field [9, 10]. In addition, in all the cases (type D spacetime: Taub-NUT, type D-Kasner, etc) in which an equation similar to the Teukolsky equation can be written down, one can study the various couplings between the spin of the perturbing field and the background parameters, i.e., spin-rotation, spin-acceleration couplings, etc., which can also be relevant in different contexts and from other points of view. We have obtained important results considering explicit applications to the Taub-NUT, Kerr-Taub-NUT, C-metric, spinning C-metric, Kasner and de Sitter spacetimes. For example in the Taub-NUT spacetime we have shown that the perturbing field acquires an effective spin which is simply related to the gravitomagnetic monopole parameter ℓ of the background [11]; in the C-metric case (uniformly accelerated black hole spacetime) we have been able to introduce a gravitational analog of the Stark effect, etc.

3.3.3. Metric perturbations in a Reissner-Nordström spacetime

We have recently solved the multiyear problem of a two-body system consisting of a ReissnerNordström black hole and a charged massive particle at rest at the first perturbative order. The expressions for the metric and the electromagnetic field, including the effects of the electromagnetically induced gravitational perturbation and of the gravitationally induced electromagnetic perturbation, have been presented in closed analytic formulas; the details are indicated in Appendix C. Motivated by our works an exact solution has then been found by Belinski and AleKseev [149] of which our solution is a linearization with respect to certain parameters.

3.3.4. Curvature and metric perturbations in de Sitter spacetime

de Sitter spacetime is a conformally flat spacetime and therefore has the Petrov type O. Due to the high number of symmetries as well as the particular simplicity of the associated metric it results as the best arena to perform curvature and metric perturbations simultaneously. In fact, taking advantage from the possibility of writing the metric in a spherically symmetric form, one can apply the well known approaches of Regge-Wheeler-Zerilli to discuss metric perturbations and translate then the results in terms of curvature. Closed analytic expressions involving Heun hypergeometric functions can be obtained; the details are scketched in appendix C.

3.3.5. Curvature perturbations due to spinning bodies on a Kerr background

A new scheme for computing dynamical evolutions and gravitational radiations for intermediate-mass-ratio inspirals (IMRIs) based on an effective one-body (EOB) dynamics plus Teukolsky perturbation theory has been recently derived by Wen-Biao Han and collaborators [169]. This research line is very promising in view of many possible applications ranging from the Post-Newtonian physics of binary systems to numerical relativity.

3.4. Cosmology

3.4.1. Mixmaster universe and the spectral index

We have recently revisited the Mixmaster dynamics in a new light, revealing a series of transitions in the complex scale invariant scalar invariant of the Weyl curvature tensor best represented by the speciality index S, which gives a 4-dimensional measure of the evolution of the spacetime independent of all the 3-dimensional gauge-dependent variables except the time used to parametrize it.

Its graph versus time with typical spikes in its real and imaginary parts corresponding to curvature wall collisions serves as a sort of electrocardiogram of the Mixmaster universe, with each such spike pair arising from a single circuit or “pulse” around the origin in the complex plane. These pulses in the speciality index seem to invariantly characterize some of the so called spike solutions in inhomogeneous cosmology and should play an important role in the current investigations of inhomogeneous Mixmaster dynamics. This interesting work will be certainly continue over the next few years.

3.4.2. Wave equations in de Sitter spacetime

Wave propagation on a de Sitter background spacetime can be considered for both the electromagnetic and the gravitational case under the preliminar choice of a gauge conditions. Usually, even in the recent literature, the discussion is limited to special gauge conditions only, like the harmonic one. Recently, some interest has been raised instead for the development of a systematic study in terms of the de Donder gauge since this is close to the Lorentz gauge of the electromagnetic case.

3.5. Exact solutions

3.5.1. Kerr-Schild ansatz revisited

We have recently presented an alternative derivation of Kerr solution by treating Kerr-Schild metrics as exact linear perturbations of Minkowski spacetime. Actually, they have been introduced as a linear superposition of the flat spacetime metric and a squared null vector field **k** multiplied by a scalar function H.

In the case of Kerr solution the vector **k** is geodesic and shearfree and it is independent of the mass parameter $\mathcal{M}$, which enters instead the definition of H linearly. Without introducing any assumption on the null congruence **k** and due to this linearity property, it is possible to solve the field equations order by order in powers of H in complete generality. The Ricci tensor turns out to consist of three different contributions: third order equations all imply that **k** must be geodesic; the latter, in turn, must be also shearfree as a consequence of first order equations, whereas the solution for H comes from second order equations. This work is discussed in Appendix E.

3.5.2. Rational metrics

We have reconsidered the general form for stationary and axisymmetric metrics which are solutions of the vacuum Einstein's equations. Using symmetry adapted coordinates and decomposing properly the 4-dimensional metric as a sum of 2 2-metrics, we have shown how certain solutions obtained long ago by using generating techniques and other involved procedures may have a simpler form with metric coefficients being rational (polynomial) functions.

This research line is just started, following an idea due to Prof. Roy P. Kerr (see the section "Kerr-Newman solution" of the present report)

4. Publications (2007 – 2011)

Refereed journals

1. Bini D., Geralico A., Ruffini R. J.
   Charged massive particle at rest in the field of a Reissner-Nordström black hole,
   Phys. Rev. D, 75, 044012, 2007.
   Abstract
   The interaction of a Reissner-Nordström black hole and a charged massive particle is studied in the framework of perturbation theory. The particle backreaction is taken into account, studying the effect of general static perturbations of the hole following the approach of Zerilli. The solutions of the combined Einstein-Maxwell equations for both perturbed gravitational and electromagnetic fields to first order of the perturbation are exactly reconstructed by summing all multipoles, and are given explicit closed form expressions. The existence of a singularity-free solution of the Einstein-Maxwell system requires that the charge-to-mass ratios of the black hole and of the particle satisfy an equilibrium condition which is in general dependent on the separation between the two bodies. If the black hole is undercritically charged (i.e. its charge-to-mass ratio is less than one), the particle must be overcritically charged, in the sense that the particle must have a charge-to-mass ratio greater than one. If the charge-to-mass ratios of the black hole and of the particle are both equal to one (so that they are both critically charged, or extreme), the equilibrium can exist for any separation distance, and the solution we find coincides with the linearization in the present context of the well-known Majumdar-Papapetrou solution for two extreme Reissner-Nordstrm black holes. In addition to these singularity-free solutions, we also analyze the corresponding solution for the problem of a massive particle at rest near a Schwarzschild black hole, exhibiting a strut singularity on the axis between the two bodies. The relations between our perturbative solutions and the corresponding exact two-body solutions belonging to the Weyl class are also discussed.

2. Bini D., Geralico A., Ruffini R. J.
   On the equilibrium of a charged massive particle in the field of a Reissner-Nordström black hole,
   Physics Letters A, vol. 360, 515-517, 2007.
   Abstract

The multiyear problem of a two-body system consisting of a Reissner-Nordström black hole and a charged massive particle at rest is here solved by an exact perturbative solution of the full EinsteinMaxwell system of equations. The expressions of the metric and of the electromagnetic field, including the effects of the electromagnetically induced gravitational perturbation and of the gravitationally induced electromagnetic perturbation, are presented in closed analytic formulas.

3. Bini D., de Felice F., Geralico A.
Strains and axial outflows in the field of a rotating black hole,
Phys. Rev. D, **76**, 047502, 2007.
Abstract
We study the behavior of an initially spherical bunch of particles accelerated along trajectories parallel to the symmetry axis of a rotating black hole. We find that, under suitable conditions, curvature and inertial strains compete to generate jetlike structures. This is a purely kinematical effect which does not account by itself for physical processes underlying the formation of jets. In our analysis a crucial role is played by the property of the electric and magnetic part of the Weyl tensor to be Lorentz invariant along the axis of symmetry in Kerr spacetime.

4. Bini D., Cherubini, C., Geralico A., Jantzen R.T.
Circular motion in accelerating black hole spacetimes,
International Journal of Modern Physics D, vol. 16, 1813-1828, 2007.
Abstract
The motion of test particles along circular orbits in the vacuum C metric is studied in the Frenet-Serret formalism. Special orbits and corresponding intrinsically defined geometrically relevant properties are selectively studied.

5. Bini D., Cherubini, C., Jantzen R.T.
The speciality index and the Lifshitz-Khalatnikov Kasner index parametrization,
Class. and Quantum Gravit., vol. 24, 5627-5636, 2007.
Abstract
The speciality index function $\mathcal{S}$ for any Petrov type I, II or D spacetime is shown to be a natural function of a single complex scalar quantity μ (natural modulo permutation symmetries). For the family of Kasner spacetimes, this quantity is a function of the Kasner indices alone which coincides with the real Lifshitz-Khalatnikov parameter u for those indices.

6. Bini D., Fortini, F., Geralico, A, Ortolan, A.
Quadrupole effects on the motion of extended bodies in Schwarzschild spacetime,

Class. and Quantum Gravity, vol. 25, 035005 (9pp), 2008.
Abstract
The motion of an extended body up to the quadrupolar structure is studied in the Schwarzschild background following Dixon's model and within certain restrictions (constant frame components for the spin and the quadrupole tensor, center of mass moving along a circular orbit, etc). We find a number of interesting situations in which deviations from the geodesic motion, due to the internal structure of the particle, can originate measurable effects. However, the standard clock effect for a pair of co/counter-rotating bodies spinning up/down is not modified by the quadrupolar structure of the particle.

7. Bini D., Cherubini, C., Geralico A., Jantzen R.T.
Physical frames along circular orbits in stationary axisymmetric spacetimes,
Gen. Rel. and Gravity, vol. 40, 985-1012, 2008.
Abstract
Three natural classes of orthonormal frames, namely Frenet-Serret, FermiWalker and parallel transported frames, exist along any timelike world line in spacetime. Their relationships are investigated for timelike circular orbits in stationary axisymmetric spacetimes, and illustrated for black hole spacetimes.

8. Bini D., Lusanna L.
Spin-rotation couplings: spinning test particles and Dirac field,
Gen. Rel. and Gravit., vol. 40, 1145-1177, 2008.
Abstract
The hypothesis of coupling between spin and rotation introduced long ago by Mashhoon is examined in the context of 1 + 3 and 3 + 1 spacetime splitting techniques, either in special or in general relativity. Its content is discussed in terms of classical (Mathisson-Papapetrou-Dixon-Souriau model) as well as quantum physics (Foldy-Wouthuysen transformation for the Dirac field in an external field), reviewing and discussing all the relevant theoretical literature concerning the existence of such effect. Some original contributions are also included.

9. Bini D., Geralico A., Ruffini R.
Charged massive particle at rest in the field of a Reissner-Nordström black hole II. Analysis of the electric field lines,
Phys. Rev. D, **77**, 064020, 2008.
Abstract
The properties of the electric field of a two-body system consisting of a Reissner-Nordstrm black hole and a charged massive particle at rest have recently been analyzed in the framework of first order perturbation theory following the standard approach of Regge, Wheeler, and Zerilli. In the present paper we complete this analysis by numerically

constructing and discussing the lines of force of the "effective" electric field of the sole particle with the subtraction of the dominant contribution of the black hole. We also give the total field due to the black hole and the particle. As the black hole becomes extreme an effect analogous to the Meissner effect arises for the electric field, with the "effective field" lines of the point charge being expelled by the outer horizon of the black hole. This effect existing at the level of test field approximation, i.e. by neglecting the backreaction on the background metric and electromagnetic field due to the particles mass and charge, is here found also at the complete perturbative level. We point out analogies with similar considerations for magnetic fields by Bicak and Dvorak. We also explicitly show that the linearization of the recently obtained Belinski-Alekseev exact solution coincides with our solution in the Regge-Wheeler gauge. Our solution thus represents a bridge between the test field solution, which neglects all the feedback terms, and the exact two-body solution, which takes into account all the nonlinearity of the interaction.

10. Bini D., Fortini, F., Geralico, A., Ortolan, A.
Quadrupole effects on the motion of extended bodies in Kerr spacetime,
Class. and Quantum Gravit., vol. 25, 125007, 2008.
Abstract
The motion of a body endowed with a dipolar as well as a quadrupolar structure is investigated in the Kerr background according to the Dixon model, extending a previous analysis done in the Schwarzschild background. The full set of evolution equations is solved under the simplifying assumptions of constant frame components for both the spin and the quadrupole tensors and that the center of mass moves along an equatorial circular orbit, the total 4-momentum of the body being aligned with it. We find that the motion deviates from the geodesic one due to the internal structure of the body, leading to measurable effects. Corrections to the geodesic value of the orbital period of a close binary system orbiting the galactic center are discussed assuming that the galactic center is a Kerr supermassive black hole.

11. Bini D., Succi S.
Analogy between capillary motion and Friedmann-Robertson-Walker cosmology,
Europhysics Letters, **82**, 34003, 2008.
Abstract
A formal equivalence between the motion of an inviscid fluid in a capillary tube and the Friedmann-Robertson-Walker cosmological equations is discussed. Similarly to the case of "sonic black holes" or "black hole analogs," largely discussed in recent literature, it is hoped that this

analogy may inspire a class of capillary-filling experiments reproducing simple cosmological scenarios in terrestrial laboratories.

12. Bini D., de Felice, F., Geralico, A.,
    Relative strains in the Kerr-Taub-NUT spacetime,
    Il Nuovo Cimento B, vol. 122, 499-504, 2007.
    Abstract
    The strains experienced by static observers in the Kerr-Taub-NUT spacetime are investigated. The role played in this context by the physical parameters characterizing this solution (i.e. the rotation parameter and NUT parameter) is discussed by analysing the limits of Kerr, Taub-NUT, NUT and Schwarzschild space-time as well as the limit of weak gravitational field.

13. Bini D., Cherubini, C., Chicone, C., Mashhoon, B.
    Gravitational induction
    Classical and Quantum Gravity, vol. 25, 225014 2008. *Abstract*
    We study the linear post-Newtonian approximation to general relativity known as gravitoelectromagnetism (GEM); in particular, we examine the similarities and differences between GEM and electrodynamics. Notwithstanding some significant differences between them, we find that a special nonstationary metric in GEM can be employed to show *explicitly* that it is possible to introduce gravitational induction within GEM in close analogy with Faraday's law of induction and Lenz's law in electrodynamics. Some of the physical implications of gravitational induction are briefly discussed.

14. Bini D., Geralico, A., Ruggiero, M. L., Tartaglia A.,
    Emission vs Fermi coordinates: applications to relativistic positioning systems
    Classical and Quantum Gravity, vol. 25, 205011 (11pp), 2008.
    Abstract
    A 4-dimensional relativistic positioning system for a general spacetime is constructed by using the so called "emission coordinates." The results apply in a small region around the world line of an accelerated observer carrying a Fermi triad, as described by the Fermi metric. In the case of a Schwarzschild spacetime modeling the gravitational field around the Earth and an observer at rest at a fixed spacetime point, these coordinates realize a relativistic positioning system alternative to the current GPS system. The latter is indeed essentially conceived as Newtonian, so that it necessarily needs taking into account at least the most important relativistic effects through Post-Newtonian corrections to work properly. Previous results concerning emission coordinates in flat spacetime are thus extended to this more general situation. Furthermore, the mapping between spacetime coordinates and emission

coordinates is completely determined by means of the world function, which in the case of a Fermi metric can be explicitly obtained.

15. Bini D., Cherubini, C., Geralico A.
Massless field perturbations of the spinning C metric,
JMP, vol. 49, 062502, 2008.
Abstract
A single master equation is given describing spin $s \leq 2$ test fields that are gauge- and tetrad-invariant perturbations of the *spinning C metric* spacetime representing a source with mass $\mathcal{M}$, uniformly rotating with angular momentum per unit mass a and uniformly accelerated with acceleration A. This equation can be separated into its radial and angular parts. The behavior of the radial functions near the black hole (outer) horizon is studied to examine the influence of A on the phenomenon of superradiance, while the angular equation leads to modified spin-weighted spheroidal harmonic solutions generalizing those of the Kerr spacetime. Finally the coupling between the spin of the perturbing field and the acceleration parameter A is discussed.

16. Bini D., Geralico A., Ruffini R.
On the linearization of the Belinski-Alekseev exact solution for two charged masses in equilibrium,
IJMPA, **23**, 1226 - 1230, 2008.
Abstract
A perturbative solution describing a two-body system consisting of a Reissner-Nordstrm black hole and a charged massive particle at rest is presented. The coincidence between such a solution and the linearized form of the recently obtained Belinski-Alekseev exact solution is explicitly shown.

17. Bini D., Capozziello S., Esposito G.
Gravitational waves about curved backgrounds: a consistency analysis in De Sitter spacetime,
International Journal of Geometric Methods in Modern Physics Vol. 5, No. 7 10691083, 2008. *Abstract*
Gravitational waves are considered as metric perturbations about a curved background metric, rather than the flat Minkowski metric since several situations of physical interest can be discussed by this generalization. In this case, when the de Donder gauge is imposed, its preservation under infinitesimal spacetime diffeomorphisms is guaranteed if and only if the associated covector is ruled by a second-order hyperbolic operator which is the classical counterpart of the ghost operator in quantum gravity. In such a wave equation, the Ricci term has opposite sign with respect to the wave equation for Maxwell theory in the Lorenz gauge. We are, nevertheless, able to relate the solutions of the two problems,

and the algorithm is applied to the case when the curved background geometry is the de Sitter spacetime. Such vector wave equations are studied in two different ways: i) an integral representation, ii) through a solution by factorization of the hyperbolic equation. The latter method is extended to the wave equation of metric perturbations in the de Sitter spacetime. This approach is a step towards a general discussion of gravitational waves in the de Sitter spacetime and might assume relevance in cosmology in order to study the stochastic background emerging from inflation.

18. Bini D., Fortini, F., Geralico, A., Ortolan, A.
Dixon's extended bodies and impulsive gravitational waves,
Physics Letters A, vol. 372, 62216225, 2008.
Abstract
The reaction of an extended body to the passage of an exact plane gravitational wave is discussed following Dixons model. The analysis performed shows several general features, e.g. even if initially absent, the body acquires a spin induced by the quadrupole structure, the center of mass moves from its initial position, as well as certain spin-flip or spin-glitch effects which are being observed.

19. Bini D., Cherubini C., Filippi S.,
On the effective geometries in classical selfgravitating systems
Phys. Rev. D, vol. 78, 064024(10 pages), 2008.
Abstract
Given a perfect barotropic and irrotational Newtonian selfgravitating fluid, perturbations with respect to a background solutions are studied. The field equations can be rearranged in a novel generalization of standard induced metric formalism which takes into account the gravitational backreaction however. The case of perturbations of polytropic spherical stars described by Lane-Emden's equation, for which the Matching problem results mathematically directly accessible, is studied in detail in the known cases of existing analytic solutions. This novel formulation presents a natural scenario in which the acoustic analogy has practical applications both for stellar and galactic dynamics.

20. Bini D., Cherubini C., Filippi S.
On vortices heating biological excitable media,
Chaos, Solitons and Fractals vol. 42, 20572066, 2009
Abstract
An extension of the Hodgkin-Huxley model for the propagation of nerve signal which takes into account dynamical heat transfer in biological tissue is derived and ne tuned with existing experimental data. The medium is heated due to the Joule's eect associated with action potential propagation, leading to characteristic thermal patterns. The intro-

duction of heat transfer—necessary on physical grounds—provides a novel way to directly observe movement of the tip of spiral waves in numerical simulations and possibly in experiments regarding biological excitable media.

21. Bini D., Jantzen R. T., Stella L.
The general relativistic Poynting-Robertson effect
Classical and Quantum Gravity, vol. 26, 055009, 2009.
Abstract
The general relativistic version is developed for Robertsons discussion of the Poynting-Robertson effect that he based on special relativity and Newtonian gravity for point radiation sources like stars. The general relativistic model uses a test radiation field of photons in outward radial motion with zero angular momentum in the equatorial plane of the exterior Schwarzschild or Kerr spacetime.

22. Bini D., Cherubini C., Geralico A., Jantzen R. T.
Electrocardiogram of the Mixmaster Universe
Classical and Quantum Gravity, vol. 26, 025012, 2009.
Abstract
The Mixmaster dynamics is revisited in a new light as revealing a series of transitions in the complex scale invariant scalar invariant of the Weyl curvature tensor best represented by the speciality index S, which gives a 4-dimensional measure of the evolution of the spacetime independent of all the 3-dimensional gauge-dependent variables except the time used to parametrize it. Its graph versus time with typical spikes in its real and imaginary parts corresponding to curvature wall collisions serves as a sort of electrocardiogram of the Mixmaster universe, with each such spike pair arising from a single circuit or pulse around the origin in the complex plane. These pulses in the speciality index seem to invariantly characterize some of the so called spike solutions in inhomogeneous cosmology and should play an important role in the current investigations of inhomogeneous Mixmaster dynamics.

23. Bini D., Cherubini, C., Geralico, A., Ortolan, A.
Dixon's extended bodies and weak gravitational waves,
General Relativity and Gravitation, vol. 41, 105, 2009. *Abstract*
General relativity considers Dixons theory as the standard theory to deal with the motion of extended bodies in a given gravitational background. We discuss here the features of the reaction of an extended body to the passage of a weak gravitational wave. We find that the body acquires a dipolar moment induced by its quadrupole structure. Furthermore, we derive the world function for the weak field limit of a gravitational wave background and use it to estimate the deviation between geodesics and the world lines of structured bodies. Measuring

such deviations, due to the existence of cumulative effects, should be favorite with respect to measuring the amplitude of the gravitational wave itself.

24. Bini D., Esposito G., Montaquila R.V.,
The vector wave equation in de Sitter space-time
General Relativity and Gravitation,, vol. 42, 51-61, 2009.
Abstract
The vector wave equation, supplemented by the Lorenz gauge condition, is decoupled and solved exactly in de Sitter space-time studied in static spherical coordinates. One component of the vector field is expressed, in its radial part, through the solution of a fourth-order ordinary differential equation obeying given initial conditions. The other components of the vector field are then found by acting with lower-order differential operators on the solution of the fourth-order equation (while the transverse part is decoupled and solved exactly from the beginning). The whole four-vector potential is eventually expressed through hypergeometric functions and spherical harmonics. Its radial part is plotted for given choices of initial conditions. This is an important step towards solving exactly the tensor wave equation in de Sitter space-time, which has important applications to the theory of gravitational waves about curved backgrounds.

25. Bini D., Geralico A., Luongo O. and Quevedo H.,
Generalized Kerr spacetime with an arbitrary mass quadrupole moment: geometric properties versus particle motion
Classical and Quantum Gravity, vol. 26, 225006 (23pp), 2009.
Abstract
An exact solution of Einstein's field equations in empty space first found in 1985 by Quevedo and Mashhoon is analyzed in detail. This solution generalizes Kerr spacetime to include the case of matter with an arbitrary mass quadrupole moment and is specified by three parameters, the mass M, the angular momentum per unit mass a and the quadrupole parameter q. It reduces to the Kerr spacetime in the limiting case q = 0 and to the ErezRosen spacetime when the specific angular momentum a vanishes. The geometrical properties of such a solution are investigated. Causality violations, directional singularities and repulsive effects occur in the region close to the source. Geodesic motion and accelerated motion are studied on the equatorial plane which, due to the reflection symmetry property of the solution, also turns out to be a geodesic plane.

26. Bini D., Cherubini C., Filippi S., Geralico A.
Extended bodies with quadrupole moment interacting with gravitational monopoles: reciprocity relations

General Relativity and Gravitation, vol. 41, 2781, 2009.
Abstract
An exact solution of Einstein's equations representing the static gravitational field of a quasi-spherical source endowed with both mass and mass quadrupole moment is considered. It belongs to the Weyl class of solutions and reduces to the Schwarzschild solution when the quadrupole moment vanishes. The geometric properties of timelike circular orbits (including geodesics) in this spacetime are investigated. Moreover, a comparison between geodesic motion in the spacetime of a quasi-spherical source and non-geodesic motion of an extended body also endowed with both mass and mass quadrupole moment as described by Dixon's model in the gravitational field of a Schwarzschild black hole is discussed. Certain "reciprocity relations" between the source and the particle parameters are obtained, providing a further argument in favor of the acceptability of Dixon's model for extended bodies in general relativity.

27. Bini D. , C. Cherubini, S. Filippi, A. Gizzi and P. E. Ricci
On the universality of spiral waves
Communications in Computational Physics (CiCP), vol. 8, pp. 610-622, 2010.
Abstract
Spiral waves appear in many different contexts: excitable biological tissues, fungi and amoebae colonies, chemical reactions, growing crystals, fluids and gas eddies as well as in galaxies. While the existing theories explain the presence of spirals in terms of nonlinear parabolic equations, in this paper it is shown that self-sustained spiral wave regime is already present in the linear heat operator, in terms of integer Bessel functions of complex argument. Such solutions, even if commonly not discussed in the literature because diverging at spatial infinity, play a central role in the understanding of the universality of spiral process. As an example we have studied how in nonlinear reaction-diffusion models the linear part of the equations determines the wave front appearance while nonlinearities are mandatory to cancel out the blowup of solutions. The spiral wave pattern still requires however at least two cross diffusing species to be physically realized.

28. Bini D., Geralico A., Kerr R.P.
The Kerr-Schild ansatz revised
International Journal of Geometric Methods in Modern Physics (IJGMMP) vol.7, 693-703, 2010.
Abstract
KerrSchild metrics have been introduced as a linear superposition of the flat spacetime metric and a squared null-vector field, say k, mul-

tiplied by some scalar function, say H. The basic assumption which led to Kerr solution was that k be both geodesic and shearfree. This condition is relaxed here and KerrSchild Ansatz is revised by treating KerrSchild metrics as exact linear perturbations of Minkowski spacetime. The scalar function H is taken as the perturbing function, so that Einsteins field equations are solved order-by-order in powers of H. It turns out that the congruence must be geodesic and shearfree as a consequence of third- and second-order equations, leading to an alternative derivation of Kerr solution.

29. Bini D. , Geralico A.
Spinning bodies and the Poynting-Robertson effect in the Schwarzschild spacetime
Classical and Quantum Gravity, vol. 27, 185014, 2010.
Abstract
A spinning particle in the Schwarzschild spacetime deviates from geodesic behavior because of its spin. A spinless particle also deviates from geodesic behavior when a test radiation field is superimposed on the Schwarzschild background: in fact the interaction with the radiation field, i.e., the absorption and re-emission of radiation, leads to a friction-like drag force responsible for the well known effect which exists already in Newtonian gravity, the Poynting-Robertson effect. Here the Poynting-Robertson effect is extended to the case of spinning particles by modifying the Mathisson-Papapetrou model describing the motion of spinning test particles to account for the contribution of the radiation force. The resulting equations are numerically integrated and some typical orbits are shown in comparison with the spinless case. Furthermore, the interplay between spin and radiation forces is discussed by analyzing the deviation from circular geodesic motion on the equatorial plane when the contribution due to the radiation can also be treated as a small perturbation. Finally the estimate of the amount of radial variation from the geodesic radius is shown to be measurable in principle.

30. Bini D. , C. Cherubini, S. Filippi, Geralico A.
Effective geometry of the $n = 1$ uniformly rotating self-gravitating polytrope
Physical Review D, vol. 82, 044005 2010.
Abstract
The "effective geometry" formalism is used to study the perturbations of a perfect barotropic Newtonian self-gravitating rotating and compressible fluid coupled with gravitational backreaction. The case of a uniformly rotating polytrope with index $n = 1$ is investigated, due to its analytical tractability. Special attention is devoted to the geometrical properties of the underlying background acoustic metric, focusing

in particular on null geodesics as well as on the analog light cone structure.

31. Bini D. , Geralico A., Jantzen R.T.
Fermi coordinates in Schwarzschild spacetime: closed form expressions
General Relativity and Gravitation, vol. 43, 18371853, 2011.
Abstract
Fermi coordinates are constructed as exact functions of the Schwarzschild coordinates around the world line of a static observer in the equatorial plane of the Schwarzschild spacetime modulo a single impact parameter determined implicitly as a function of the latter coordinates. This illustrates the difficulty of constructing explicit exact Fermi coordinates even along simple world lines in highly symmetric spacetimes.

32. Bini D., Geralico A., Jantzen R. T.
Spin-geodesic deviations in the Schwarzschild spacetime
General Relativity and Gravitation, vol. 43, 959-975, 2011.
Abstract
The deviation of the path of a spinning particle from a circular geodesic in the Schwarzschild spacetime is studied by an extension of the idea of geodesic deviation. Within the Mathisson-Papapetrou-Dixon model and assuming the spin parameter to be sufficiently small so that it makes sense to linearize the equations of motion in the spin variables as well as in the geodesic deviation, the spin-curvature force adds an additional driving term to the second order system of linear ordinary differential equations satisfied by nearby geodesics. Choosing initial conditions for geodesic motion leads to solutions for which the deviations are entirely due to the spin-curvature force, and one finds that the spinning particle position for a given fixed total spin oscillates roughly within an ellipse in the plane perpendicular to the motion, while the azimuthal motion undergoes similar oscillations plus an additional secular drift which varies with spin orientation.

33. Gizzi A., Bernaschi M., Bini D., Cherubini C., Filippi S., Melchionna S., Succi S.
Three-band decomposition analysis of wall shear stress in pulsatile flows
Physical Review E **83**, 031902(10), 2011.
Abstract
Space-time patterns of Wall Shear Stress (WSS) resulting from the numerical simulation of pulsating hemodynamic flows in semi-coronal domains are analyzed, both in the case of regular semi-coronal domains and semi-coronal domains with bumpy insertions, mimicking aneurysm-like geometries. A new family of cardiovascular risk indicators, which we name Three-Band Diagrams (TBD), are introduced, as a sensible

generalization of the two standard indicators, i.e. the time-averaged WSS and the OSI (Oscillatory Shear Index). TBD's provide a handy access to additional information contained in the dynamic structure of the WSS signal as a function of the physiological WSS risk threshold, thereby allowing a quick visual assessment of the risk sensitivity to individual fluctuations of the physiological risk thresholds. Due to its generality, TBD analysis is expected to prove useful for a wide host of applications in science, engineering and medicine, where risk-assessment plays a central role.

34. Bini D., Geralico A., Jantzen R. T., Semeřák O. and Stella L.
The general relativistic Poynting-Robertson effect II: A photon flux with nonzero angular momentum
Classical and Quantum Gravity, vol. 28 035008 (21pp), 2011.
Abstract
We study the motion of a test particle in a stationary, axially and reflection symmetric spacetime of a central compact object, as affected by interaction with a test radiation field of the same symmetries. Considering the radiation flux with fixed but arbitrary (non-zero) angular momentum, we extend previous results limited to an equatorial motion within a zero-angular-momentum photon flux in the Kerr and Schwarzschild backgrounds. While a unique equilibrium circular orbit exists if the photon flux has zero angular momentum, multiple such orbits appear if the photon angular momentum is sufficiently high.

35. Bini D., Cherubini C., Filippi S.
Effective geometry of a white dwarf
Physical Review D, vol. 83, 064039 (15pp), 2011.
Abstract
The "effective geometry" formalism is used to study the perturbations of a white dwarf described as a self-gravitating fermion gas with a completely degenerate relativistic equation of state of barotropic type. The quantum nature of the system causes an absence of homological properties manifested instead by batropic stars and requires a parametric study of the solutions both at numerical and analytical level. We have explicitly derived a compact analytical parametric approximate solution of Padé type which gives density curves and stellar radii in good accordance with already existing numerical results. After validation of this new type of approximate solutions, we use them to construct the effective acoustic metric governing perturbations of any type following Chebsch's formalism. Even in this quantum and relativistic case the stellar surface exhibits a curvature singularity due to the vanishing of density, as already evidenced in past studies on non relativistic and non quantum self-gravitating polytropic star. The equations of the the-

ory are finally numerically integrated, in the simpler case of irrotational spherical pulsating perturbations including the effect of back-reaction, in order to have a dynamical picture of the process occurring in the acoustic metric.

36. Bini D. , de Felice F., Geralico A.
Accelerated orbits in black hole fields: the static case
Classical and Quantum Gravity, vol. 28 225012, 2011.
Abstract
We study non-geodesic orbits of test particles endowed with a structure, assuming the Schwarzschild spacetime as background. We develop a formalism which allows one to recognize the geometrical characterization of those orbits in terms of their Frenet-Serret parameters and apply it to explicit cases as those of spatially circular orbits which witness the equilibrium under conflicting types of interactions. In our general analysis we solve the equations of motion offering a detailed picture of the dynamics having in mind a check with a possible astronomical set up. We focus on certain ambiguities which plague the interpretation of the measurements preventing one from identifying the particular structure carried by the particle.

37. Bini D., Esposito G., Geralico A.
de Sitter spacetime: effects of metric perturbations on geodesic motion
General Relativity and Gravitation, to appear, 2011.
Abstract
Gravitational perturbations of the de Sitter spacetime are investigated using the Regge–Wheeler formalism. The set of perturbation equations is reduced to a single second order differential equation of the Heun-type for both electric and magnetic multipoles. The solution so obtained is used to study the deviation from an initially radial geodesic due to the perturbation. The spectral properties of the perturbed metric are also analyzed. Finally, gauge- and tetrad-invariant first-order massless perturbations of any spin are explored following the approach of Teukolsky. The existence of closed-form, i.e. Liouvillian, solutions to the radial part of the Teukolsky master equation is discussed.

38. Bini D., Geralico A., Jantzen R. T. and Semeřák O.
Effect of radiation flux on test particle motion in the Vaidya spacetime
Classical and Quantum Gravity, to appear, 2011.
Abstract
Motion of massive test particles in the nonvacuum spherically symmetric radiating Vaidya spacetime is investigated, allowing for physical interaction of the particles with the radiation field in terms of which the source energy-momentum tensor is interpreted. This "Poynting-Robertson-like effect" is modeled by the usual effective term describing

a Thomson-type radiation drag force. The equations of motion are studied for simple types of motion including free motion (without interaction), purely radial and purely azimuthal (circular) motion, and for the particular case of "static" equilibrium; appropriate solutions are given where possible. The results—mainly those on the possible existence of equilibrium positions—are compared with their counterparts obtained previously for a test spherically symmetric radiation field in a vacuum Schwarzschild background.

39. Bini D. and Geralico A.
Spin-geodesic deviations in the Kerr spacetime
Physical Review D, to appear, 2011.
Abstract
The dynamics of extended spinning bodies in the Kerr spacetime is investigated by assuming that the actual motion slightly deviates from the geodesic path due to the spin-curvature force, in order to focus on how the presence of the spin changes that geodesic motion. As usual, the spin parameter is assumed to be very small in order to neglect the back reaction on the spacetime geometry. This approach naturally leads to solve the Mathisson-Papapetrou-Dixon equations linearized in the spin variables as well as in the deviation vector, with the same initial conditions as for geodesic motion. General deviations from generic geodesic motion are studied, generalizing previous results limited to the very special case of an equatorial circular geodesic as the reference path.

40. Bini D., Geralico A., Jantzen R. T.
Separable geodesic action slicing in stationary spacetimes
General Relativity and Gravitation, to appear, 2011.
Abstract
A simple observation about the action for geodesics in a stationary spacetime with separable geodesic equations leads to a natural class of slicings of that spacetime whose orthogonal geodesic trajectories represent the world lines of freely falling fiducial observers. The time coordinate function can then be taken to be the observer proper time, leading to a unit lapse function, although the time coordinate lines still follow Killing trajectories to retain the explicitly stationary nature of the coordinate grid. This explains some of the properties of the original Painlevé-Gullstrand coordinates on the Schwarzschild spacetime and their generalization to the Kerr-Newman family of spacetimes, reproducible also locally for the Gödel spacetime. For the static spherically symmetric case the slicing can be chosen to be intrinsically flat with spherically symmetric geodesic observers, leaving all the gravitational field information in the shift vector field.

41. Bini D., Fortini P., Haney M., Ortolan A.

Electromagnetic waves in gravitational wave spacetimes
Classical and Quantum Gravity, vol. 28, 235007, 2011.
Abstract
We have considered the propagation of electromagnetic waves in a spacetime representing an exact gravitational plane wave and calculated the induced changes on the four potential field A^μ of a plane electromagnetic wave. By choosing a suitable photon round-trip in a Michelson interferometer, we have been able to identify the physical effects of the exact gravitational wave on the electromagnetic field, i.e. phase shift, change of the polarization vector, angular deflection and delay. These results have been exploited to study the response of an interferometric gravitational wave detector beyond the linear approximation of the general theory of relativity.

42. Bini D. and Geralico A.
Scattering by an electromagnetic radiation field
submitted, 2011.
Abstract
Motion of test particles in the gravitational field associated with an electromagnetic plane wave is investigated. The interaction with the radiation field is modeled by a force term *à la* Poynting-Robertson entering the equations of motion given by the 4-momentum density of radiation observed in the particle's rest frame with a multiplicative constant factor expressing the strength of the interaction itself. Explicit analytical solutions are obtained. Scattering of fields by the electromagnetic wave, i.e., scalar (spin 0), massless spin $\frac{1}{2}$ and electromagnetic (spin 1) fields, is studied too.

43. Bini D., Gregoris D. and Succi S.
Kinetic theory in a curved spacetime: applications to the Poynting-Robertson effect
submitted, 2011.
Abstract
We discuss the statistical description of a massive and a photon gas in an arbitrary spacetime. The motion of a massive test particle inside a (test) photon gas is then studied near a Schwarzschild black hole, leading to a novel description of radiation scattering in terms of the so called Poynting-Robertson effect. Based on the new statistical description, and at variance with previous results, it is found that a particle moving in the gravitational background of a Schwarzschild black-hole, always ends up within the black-hole horizon.

Books and book chapters

1. (**Chapter in Book)** Ferrarese G. and Bini D. ,
Compatibility of physical frames in relativity,

Rendiconti del Circolo Matematico di Palermo, serie II, Suppl. 78, 97-110, 2006.
Abstract
Certain notions concerning physical frames thought as geometrical support of continuous systems are discussed; from these notions, independently from the continuum dynamics, the Cauchy problem for the *first order characteristics of the frame*, as well as the associated (involutive) *compatibility conditions*, *involving only the initial data*, are considered.

2. (**Book)** G. Ferrarese, Bini D.
   Introduction to relativistic continuum mechanics,
   Lecture Notes in Physics **727**, Ed. Springer, 2007.

3. (**Book)** De Felice F., Bini D.
   Classical Measurements in Curved Space-Times
   Series: Cambridge Monographs on Mathematical Physics, Cambridge, UK, 2010

   Brief description

   The theory of relativity describes the laws of physics in a given space-time. However, a physical theory must provide observational predictions expressed in terms of measurements, which are the outcome of practical experiments and observations. Ideal for readers with a mathematical background and a basic knowledge of relativity, this book will help readers understand the physics behind the mathematical formalism of the theory of relativity. It explores the informative power of the theory of relativity, and highlights its uses in space physics, astrophysics and cosmology. Readers are given the tools to pick out from the mathematical formalism those quantities that have physical meaning and which can therefore be the result of a measurement. The book considers the complications that arise through the interpretation of a measurement, which is dependent on the observer who performs it. Specific examples of this are given to highlight the awkwardness of the problem.
   Provides a large sample of observers and reference frames in space-times that can be applied to space physics, astrophysics and cosmology. Tackles the problems encountered in interpreting measurements, giving specific examples. Features advice to help readers understand the logic of a given theory and its limitations.

   Contents
   1. Introduction; 2. The theory of relativity: a mathematical overview; 3. Space-time splitting; 4. Special frames; 5. The world function; 6. Local measurements; 7. Non-local measurements; 8. Observers in physical

relevant space-times; 9. Measurements in physically relevant space-times; 10. Measurements of spinning bodies.

APPENDICES

A. Spacetime splitting techniques in general relativity

The concept of a "gravitational force" modeled after the electromagnetic Lorentz force was born in the Newtonian context of centrifugal and Coriolis "fictitious" forces introduced by a rigidly rotating coordinate system in a flat Euclidean space. Bringing this idea first into linearized general relativity and then into its fully nonlinear form, it has found a number of closely related but distinct generalizations. Regardless of the details, this analogy between gravitation and electromagnetism has proven useful in interpreting the results of spacetime geometry in terms we can relate to, and has been illustrated in many research articles and textbooks over the past half century.

ICRANet has itself devoted a workshop and its proceedings to aspects of this topic in 2003 [2]. In the lengthy introduction to these proceedings, R. Ruffini has discussed a number of related topics, like "the gravitational analogue of the Coulomb-like interactions, of Hertz-like wave solutions, of the Oersted-Ampére-like magnetic interaction, etc.," supporting the thesis that treating gravitation in analogy with electromagnetism may help to better understand the main features of certain gravitational phenomena, at least when the gravitational field may be considered appropriately described by its linearized approximation [12, 13, 14, 15, 16, 17, 18]. A particularly long bibliography surveying most of the relevant literature through 2001 had been published earlier in the Proceedings of one of the annual Spanish Relativity Meetings [19].

In the 1990s, working in fully nonlinear general relativity, all of the various notions of "noninertial forces" (centrifugal and Coriolis forces) were put into a single framework by means of a unifying formalism dubbed "gravitoelectromagnetism" [1, 3, 4] which is a convenient framework to deal with these and curvature forces and related questions of their effect on test bodies moving in the gravitational field. More precisely, such a language is based on the splitting of spacetime into "space plus time," accomplished locally by means of an observer congruence, namely a congruence of timelike worldlines with (future-pointing) unit tangent vector field u which may be interpreted as the 4-velocity field of a family of test observers filling some region of spacetime. The orthogonal decomposition of each tangent space into a local time direction along u and the orthogonal local rest space (LRS) is used to decompose all spacetime tensors and tensor equations into a "space plus time" representation; the latter representation is somehow equivalent to a geometrical "measurement" process. This leads to a family of "spatial" spacetime tensor fields which represent each spacetime field and a family of spatial equations which represent each spacetime equation. Dealing with spacetime splitting techniques as well as 3-dimensional-like quantities clearly permits a better interface of our intuition and experience with the 4-dimensional geometry in certain gravitational problems. It can be particularly useful in spacetimes which have a geometrically defined timelike congruence, either explicitly given or defined implicitly as the congruence of orthogonal trajectories to a slicing or foliation of spacetime by a family of privileged spacelike hypersurfaces.

For example, splitting techniques are useful in the following spacetimes:

1. Stationary spacetimes, having a preferred congruence of Killing trajectories associated with the stationary symmetry, which is timelike on a certain region of spacetime (usually an open region, the boundary of which corresponding to the case in which the Killing vector becomes null so that in the exterior region Killing trajectories are spacelike).

2. Stationary axially symmetric spacetimes having in addition a preferred slicing whose orthogonal trajectories coincide with the worldlines of locally nonrotating test observers.

3. Cosmological spacetimes with a spatial homogeneity subgroup, which have a preferred spacelike slicing by the orbits of this subgroup.

From the various schools of relativity that blossomed during the second half of the last century a number of different approaches to spacetime splitting were developed without reference to each other. During the 1950s efforts were initiated to better understand general relativity and the mathematical tools needed to flush out its consequences. Lifshitz and the Russian school, Lichnerowicz in France, the British school, scattered Europeans (Ehlers and Trautman, for example) and the Americans best represented by Wheeler initiated this wave of relativity which blossomed in the 1960s. The textbook of Landau and Lifshitz and articles of Zelmanov [20, 21, 22, 23] presented the "threading point of view" of the Russian school and of Moller [21] which influenced Cattaneo in Rome and his successor Ferrarese [24, 25, 26, 27, 28], while a variation of this approach not relying on a complementary family of hypersurfaces (the "congruence point of view) began from work initially codified by Ehlers [17] and then taken up by Ellis [29, 30] in analyzing cosmological issues.

However, issues of quantum gravity lead to the higher profile of the "slicing point of view" in the 1960s initiated earlier by Lichnerowicz and developed by Arnowitt, Deser and Misner and later promoted by the influential textbook "Gravitation" by Misner, Thorne and Wheeler [31, 32, 33, 34] represents a splitting technique which is complementary to the threading point of view and its congruence variation, and proved quite useful in illuminating properties of black hole spacetimes.

R. Ruffini, a former student of Cattaneo and a collaborator of Wheeler, in his quest to better understand electromagnetic properties of black holes, awakened the curiosity of Jantzen and Carini at the end of the 1980s, later joined by Bini, who together made an effort to clarify the interrelationships between these various approaches as well as shed some light on the then confusing work of Abramowicz and others on relativistic centrifugal and Coriolis forces. By putting them all in a common framework, and describing what each measured in geometrical terms, and how each related to the others, some order was brought to the field [1, 3, 4].

The ICRANet people working on this subject have applied the main ideas underlying spacetime splitting techniques to concrete problems arising when studying test particle motion in black hole spacetimes. Among the various results obtained it is worth mentioning the relativistic and geometrically correct definition of inertial forces in general relativity [35, 36, 37, 38, 39], the definition of special world line congruences, relevant for the description of the motion of test particles along circular orbits in the Kerr spacetime (geodesic meeting point observers, extremely accelerated observers, etc.), the specification of all the geometrical properties concerning observer-adapted frames to the above mentioned special world line congruences [40, 41], the characterization of certain relevant tensors in black hole spacetimes (Simon tensor, Killing-Yano tensor) in terms of gravitoelectromagnetism [42, 43], etc. This research line is still ongoing and productive.

Over a period of several decades Jantzen, Bini and a number of students at the University of Rome "La Sapienza" under the umbrella of the Rome ICRA group have been working on this problem under the supervision of Ruffini. The collaborators involved have been already listed and the most relevant papers produced are indicated in the references below [44]–[83]. In the present year 2010 a book by F. de Felice and D. Bini, including a detailed discussion of this and related topics, has been published by Cambridge University Press [157].

Let us now describe some fundamental notions of gravitoelectromagnetism.

A.1. Observer-orthogonal splitting

Let ${}^{(4)}$g (signature −+++ and components ${}^{(4)}g_{\alpha\beta}$, $\alpha, \beta, \ldots = 0, 1, 2, 3$) be the spacetime metric, ${}^{(4)}\nabla$ its associated covariant derivative operator, and ${}^{(4)}\eta$ the unit volume 4-form which orients spacetime (${}^{(4)}\eta_{0123} = {}^{(4)}g^{1/2}$ in an oriented frame, where ${}^{(4)}g \equiv |\det({}^{(4)}g_{\alpha\beta})|$). Assume the spacetime is also time oriented and let u be a future-pointing unit timelike vector field ($u^\alpha u_\alpha = -1$) representing the 4-velocity field of a family of test observers filling the spacetime (or some open submanifold of it).

If S is an arbitrary tensor field, let $S^\flat$ and $S^\sharp$ denote its totally covariant and totally contravariant forms with respect to the metric index-shifting operations. It is also convenient to introduce the right contraction notation $[S \llcorner X]^\alpha = S^\alpha{}_\beta X^\beta$ for the contraction of a vector field and the covariant index of a $\binom{1}{1}$-tensor field (left contraction notation being analogous).

A.1.1. The measurement process

The observer-orthogonal decomposition of the tangent space, and in turn of the algebra of spacetime tensor fields, is accomplished by the temporal projection operator $T(u)$ along u and the spatial projection operator $P(u)$ onto LRS_u, which may be identified with mixed second rank tensors acting by contraction

$$\begin{aligned} \delta^\alpha{}_\beta &= T(u)^\alpha{}_\beta + P(u)^\alpha{}_\beta \,, \\ T(u)^\alpha{}_\beta &= -u^\alpha u_\beta \,, \\ P(u)^\alpha{}_\beta &= \delta^\alpha{}_\beta + u^\alpha u_\beta \,. \end{aligned} \tag{A.1.1}$$

These satisfy the usual orthogonal projection relations $P(u)^2 = P(u)$, $T(u)^2 = T(u)$, and $T(u) \llcorner P(u) = P(u) \llcorner T(u) = 0$. Let

$$[P(u)S]^{\alpha\ldots}{}_{\beta\ldots} = P(u)^\alpha{}_\gamma \cdots P(u)^\delta{}_\beta \cdots S^{\gamma\ldots}{}_{\delta\ldots} \tag{A.1.2}$$

denote the spatial projection of a tensor S on all indices.

The *measurement of* S by the observer congruence is the family of spatial tensor fields which result from the spatial projection of all possible contractions of S by any number of factors of u. For example, if S is a $\binom{1}{1}$-tensor, then its measurement

$$S^\alpha{}_\beta \leftrightarrow (\underbrace{u^\delta u_\gamma S^\gamma{}_\delta}_{scalar}, \underbrace{P(u)^\alpha{}_\gamma u^\delta S^\gamma{}_\delta}_{vector}, \underbrace{P(u)^\delta{}_\alpha u_\gamma S^\gamma{}_\delta}_{vector}, \underbrace{P(u)^\alpha{}_\gamma P(u)^\delta{}_\beta S^\gamma{}_\delta}_{tensor}) \tag{A.1.3}$$

results in a scalar field, a spatial vector field, a spatial 1-form and a spatial $\binom{1}{1}$-tensor field. It is exactly this family of fields which occur in the orthogonal "decomposition of S" with respect to the observer congruence

$$\begin{aligned} S^\alpha{}_\beta &= [T(u)^\alpha{}_\gamma + P(u)^\alpha{}_\gamma][T(u)^\delta{}_\beta + P(u)^\delta{}_\beta] S^\gamma{}_\delta \\ &= [u^\delta u_\gamma S^\gamma{}_\delta] u^\alpha u_\beta + \cdots + [P(u)S]^\alpha{}_\beta \,. \end{aligned} \tag{A.1.4}$$

A.2. Examples

1. Measurement of the spacetime metric and volume 4-form

   - spatial metric $\qquad [P(u)^{(4)}\mathrm{g}]_{\alpha\beta} = P(u)_{\alpha\beta}$

- spatial unit volume 3-form $\eta(u)_{\alpha\beta\gamma} = u^{\delta}\,{}^{(4)}\eta_{\delta\alpha\beta\gamma}$;
  In a compact notation: $\eta(u) = [P(u)\, u \,\lrcorner\, {}^{(4)}\eta]$

2. Measurement of the Lie, exterior and covariant derivative
   - spatial Lie derivative $\pounds(u)_X = P(u)\pounds_X$
   - the spatial exterior derivative $d(u) = P(u)d$
   - the spatial covariant derivative $\nabla(u) = P(u)^{(4)}\nabla$
   - the spatial Fermi-Walker derivative (or Fermi-Walker temporal derivative) $\nabla_{(\mathrm{fw})}(u) = P(u)^{(4)}\nabla_u$ (when acting on spatial fields)
   - the Lie temporal derivative $\nabla_{(\mathrm{lie})}(u) = P(u)\pounds_u = \pounds(u)_u$

   Note that spatial differential operators do not obey the usual product rules for nonspatial fields since undifferentiated factors of u are killed by the spatial projection.

3. Notation for 3-dimensional operations

   It is convenient to introduce 3-dimensional vector notation for the spatial inner product and spatial cross product of two spatial vector fields X and Y. The inner product is just

$$X \cdot_u Y = P(u)_{\alpha\beta} X^\alpha Y^\beta \tag{A.2.1}$$

   while the cross product is

$$[X \times_u Y]^\alpha = \eta(u)^\alpha{}_{\beta\gamma} X^\beta Y^\gamma \,. \tag{A.2.2}$$

   With the "vector derivative operator" $\nabla(u)^\alpha$ one can introduce spatial gradient, curl and divergence operators for functions f and spatial vector fields X by

$$\begin{aligned} \mathrm{grad}_u\, f &= \nabla(u) f = [d(u) f]^\sharp \,, \\ \mathrm{curl}_u\, X &= \nabla(u) \times_u X = [{}^{*(u)} d(u) X^\flat]^\sharp \,, \\ \mathrm{div}_u\, X &= \nabla(u) \cdot_u X = {}^{*(u)}[d(u)\, {}^{*(u)} X^\flat] \,, \end{aligned} \tag{A.2.3}$$

   where ${}^{*(u)}$ is the spatial duality operation for antisymmetric tensor fields associated with the spatial volume form $\eta(u)$ in the usual way. These definitions enable one to mimic all the usual formulas of 3-dimensional vector analysis. For example, the spatial exterior derivative formula for the curl has the index form

$$[\mathrm{curl}_u\, X]^\alpha = \eta(u)^{\alpha\beta\gamma}\, {}^{(4)}\nabla_\beta X_\gamma \tag{A.2.4}$$

   which also defines a useful operator for nonspatial vector fields X.

4. Measurement of the covariant derivative of the observer four velocity

Measurement of the covariant derivative $[{}^{(4)}\nabla u]^\alpha{}_\beta = u^\alpha{}_{;\beta}$ leads to two spatial fields, the acceleration vector field $a(u)$ and the kinematical mixed tensor field $k(u)$

$$\begin{aligned} u^\alpha{}_{;\beta} &= -a(u)^\alpha u_\beta - k(u)^\alpha{}_\beta \,, \\ a(u) &= \nabla_{(\mathrm{fw})}(u) u \,, \\ k(u) &= -\nabla(u) u \,. \end{aligned} \tag{A.2.5}$$

The kinematical tensor field may be decomposed into its antisymmetric and symmetric parts:

$$k(u) = \omega(u) - \theta(u) \,, \tag{A.2.6}$$

with

$$\begin{aligned} [\omega(u)^\flat]_{\alpha\beta} &= P(u)^\sigma_\alpha P(u)^\delta_\beta u_{[\delta;\sigma]} \\ &= \tfrac{1}{2}[d(u) u^\flat]_{\alpha\beta} \,, \\ [\theta(u)^\flat]_{\alpha\beta} &= P(u)^\sigma_\alpha P(u)^\delta_\beta u_{(\delta;\sigma)} \\ &= \tfrac{1}{2}[\nabla_{(\mathrm{lie})}(u) P(u)^\flat]_{\alpha\beta} = \tfrac{1}{2}\pounds(u) u^{(4)} g_{\alpha\beta} \,, \end{aligned} \tag{A.2.7}$$

defining the mixed rotation or vorticity tensor field $\omega(u)$ (whose sign depends on convention) and the mixed expansion tensor field $\theta(u)$, the latter of which may itself be decomposed into its tracefree and pure trace parts

$$\theta(u) = \sigma(u) + \frac{1}{3}\Theta(u) P(u) \,, \tag{A.2.8}$$

where the mixed shear tensor field $\sigma(u)$ is tracefree ($\sigma(u)^\alpha{}_\alpha = 0$) and the expansion scalar is

$$\Theta(u) = u^\alpha{}_{;\alpha} = {}^{*(u)}[\nabla_{(\mathrm{lie})}(u)\eta(u)] \,. \tag{A.2.9}$$

Define also the rotation or vorticity vector field $\omega(u) = \frac{1}{2}\,\mathrm{curl}_u\, u$ as the spatial dual of the spatial rotation tensor field

$$\omega(u)^\alpha = \tfrac{1}{2}\eta(u)^{\alpha\beta\gamma}\omega(u)_{\beta\gamma} = \tfrac{1}{2}{}^{(4)}\eta^{\alpha\beta\gamma\delta} u_\beta u_{\gamma;\delta} \,. \tag{A.2.10}$$

5. Lie, Fermi-Walker and co-Fermi-Walker derivatives

The kinematical tensor describes the difference between the Lie and Fermi-Walker temporal derivative operators when acting on spatial tensor fields. For example, for a spatial vector field X

$$\begin{aligned}\nabla_{(\mathrm{fw})}(u)X^\alpha &= \nabla_{(\mathrm{lie})}(u)X^\alpha - k(u)^\alpha{}_\beta X^\beta \\ &= \nabla_{(\mathrm{lie})}(u)X^\alpha - \omega(u)^\alpha{}_\beta X^\beta + \theta(u)^\alpha{}_\beta X^\beta \,,\end{aligned} \qquad \text{(A.2.11)}$$

where

$$\omega(u)^\alpha{}_\beta X^\beta = -\eta(u)^\alpha{}_{\beta\gamma}\omega(u)^\beta X^\gamma = -[\omega(u) \times_u X]^\alpha \,. \qquad \text{(A.2.12)}$$

The kinematical quantities associated with u may be used to introduce two spacetime temporal derivatives, the Fermi-Walker derivative and the co-rotating Fermi-Walker derivative along u

$$\begin{aligned}{}^{(4)}\nabla_{(\mathrm{fw})}(u)X^\alpha &= {}^{(4)}\nabla_u X^\alpha + [a(u) \wedge u]^{\alpha\beta} X_\beta \,, \\ {}^{(4)}\nabla_{(\mathrm{cfw})}(u)X^\alpha &= {}^{(4)}\nabla_{(\mathrm{fw})}(u)X^\alpha + \omega(u)^\alpha{}_\beta X^\beta \,.\end{aligned} \qquad \text{(A.2.13)}$$

These may be extended to arbitrary tensor fields in the usual way (so that they commute with contraction and tensor products) and they both commute with index shifting with respect to the metric and with duality operations on antisymmetric tensor fields since both ${}^{(4)}g$ and ${}^{(4)}\eta$ have zero derivative with respect to both operators (as does u itself). For an arbitrary vector field X the following relations hold

$$\begin{aligned}\pounds_u X^\alpha &= {}^{(4)}\nabla_{(\mathrm{fw})}(u)X^\alpha + [\omega(u)^\alpha{}_\beta - \theta(u)^\alpha{}_\beta + u^\alpha a(u)_\beta] X^\beta \\ &= {}^{(4)}\nabla_{(\mathrm{cfw})}(u)X^\alpha + [-\theta(u)^\alpha{}_\beta + u^\alpha a(u)_\beta] X^\beta \,.\end{aligned} \qquad \text{(A.2.14)}$$

A spatial co-rotating Fermi-Walker derivative $\nabla_{(\mathrm{cfw})}(u)$ ("co-rotating Fermi-Walker temporal derivative") may be defined in a way analogous to the ordinary one, such that the three temporal derivatives have the following relation when acting on a spatial vector field X

$$\begin{aligned}\nabla_{(\mathrm{cfw})}(u)X^\alpha &= \nabla_{(\mathrm{fw})}(u)X^\alpha + \omega(u)^\alpha{}_\beta X^\beta \\ &= \nabla_{(\mathrm{lie})}(u)X^\alpha + \theta(u)^\alpha{}_\beta X^\beta \,,\end{aligned} \qquad \text{(A.2.15)}$$

while $\nabla_{(\mathrm{cfw})}(u)[fu] = fa(u)$ determines its action on nonspatial fields. It has been introduced an index notation to handle these three operators simultaneously

$$\{\nabla_{(\mathrm{tem})}(u)\}_{\mathrm{tem=fw,cfw,lie}} = \{\nabla_{(\mathrm{fw})}(u), \nabla_{(\mathrm{cfw})}(u), \nabla_{(\mathrm{lie})}(u)\} \,. \qquad \text{(A.2.16)}$$

A.3. Comparing measurements by two observers in relative motion

Suppose U is another unit timelike vector field representing a different family of test observers. One can then consider relating the "observations" of each to the other. Their relative velocities are defined by

$$\begin{aligned} U &= \gamma(U,u)[u+\nu(U,u)]\ , \\ u &= \gamma(u,U)[U+\nu(u,U)]\ , \end{aligned} \tag{A.3.1}$$

where the relative velocity $\nu(U,u)$ of U with respect to u is spatial with respect to u and vice versa, both of which have the same magnitude $||\nu(U,u)|| = [\nu(U,u)_\alpha \nu(U,u)^\alpha]^{1/2}$, while the common gamma factor is related to that magnitude by

$$\gamma(U,u) = \gamma(u,U) = [1 - ||\nu(U,u)||^2]^{-1/2} = -U_\alpha u^\alpha\ . \tag{A.3.2}$$

Let $\hat{\nu}(U,u)$ be the unit vector giving the direction of the relative velocity $\nu(U,u)$. In addition to the natural parametrization of the worldlines of U by the proper time τ_U, one may introduce two new parametrizations: by a (Cattaneo) relative standard time $\tau_{(U,u)}$

$$d\tau_{(U,u)}/d\tau_U = \gamma(U,u)\ , \tag{A.3.3}$$

which corresponds to the sequence of proper times of the family of observers from the u congruence which cross paths with a given worldline of the U congruence, and by a relative standard lenght $\ell_{(U,u)}$

$$d\ell_{(U,u)}/d\tau_U = \gamma(U,u)||\nu(U,u)|| = ||\nu(U,u)||d\tau_{(U,u)}/d\tau_U\ , \tag{A.3.4}$$

which corresponds to the spatial arc lenght along U as observed by u.

Eqs. (A.3.1) describe a unique active "relative observer boost" $B(U,u)$ in the "relative observer plane" spanned by u and U such that

$$B(U,u)u = U\ , \quad B(U,u)\nu(U,u) = -\nu(u,U) \tag{A.3.5}$$

and which acts as the identity on the common subspace of the local rest spaces $LRS_u \cap LRS_U$ orthogonal to the direction of motion.

A.3.1. Maps between the LRSs of different observers

The projection $P(U)$ restricts to an invertible map when combined with $P(u)$ as follows

$$P(U,u) = P(U) \circ P(u) : LRS_u \to LRS_U \tag{A.3.6}$$

with inverse $P(U,u)^{-1} : LRS_U \to LRS_u$ and vice versa, and these maps also act as the identity on the common subspace of the local rest spaces.

Similarly the boost $B(U,u)$ restricts to an invertible map

$$B_{(\mathrm{lrs})}(U,u) \equiv P(U) \circ B(U,u) \circ P(u) \tag{A.3.7}$$

between the local rest spaces which also acts as the identity on their common subspace. The boosts and projections between the local rest spaces differ only by a gamma factor along the direction of motion.

An expression for the inverse projection

If $Y \in LRS_u$, then the orthogonality condition $0 = u_\alpha Y^\alpha$ implies that Y has the form

$$Y = [\nu(u,U) \cdot_U P(U,u)Y]U + P(U,u)Y\,. \tag{A.3.8}$$

If $X = P(U,u)Y \in LRS_U$ is the field seen by U, then $Y = P(U,u)^{-1}X$ and

$$P(U,u)^{-1}X = [\nu(u,U) \cdot_U X]U + X = [P(U) + U \otimes \nu(u,U)^\flat] \llcorner X\,, \tag{A.3.9}$$

which gives a useful expression for the inverse projection.

This map appears in the transformation law for the electric and magnetic fields:

$$\begin{aligned} E(u) &= \gamma P(U,u)^{-1}[E(U) + \nu(u,U) \times_U B(U)]\,,\\ B(u) &= \gamma P(U,u)^{-1}[B(U) - \nu(u,U) \times_U E(U)]\,. \end{aligned} \tag{A.3.10}$$

A.4. Comparing measurements by three or more observers in relative motion

A typical situation is that of a fluid/particle whis is observed by two diferrent families of observers. In this case one deal with three timelike congruences (or two congruences and a single line): the rest frame of the fluid U and the two observer families u e u'.

All the previous formalism can be easily generalized. One has

$$\begin{aligned} U &= \gamma(U,u)[u + \nu(U,u)]\,,\\ U &= \gamma(U,u')[u' + \nu(U,u')]\,,\\ u' &= \gamma(u',u)[u + \nu(u',u)]\,,\\ u &= \gamma(u,u')[u' + \nu(u,u')]\,. \end{aligned} \tag{A.4.1}$$

and mixed projectors involving the various four-velocities can be introduced. They are summarized in the following table:

PROJECTORS	
$P(u,U,u)$	$P(u)+\gamma(U,u)^2\nu(U,u)\otimes\nu(U,u)$
$P(u,U,u)^{-1}$	$P(u)-\nu(U,u)\otimes\nu(U,u)$
$P(u,U,u')$	$P(u,u')+\gamma(U,u)\gamma(U,u')\nu(U,u)\otimes\nu(U,u')$
$P(u,U,u')^{-1}$	$P(u',u)+\gamma(u,u')[(\nu(u,u')-\nu(U,u'))\otimes\nu(U,u)$ $+\nu(U,u')\otimes\nu(u',u)]$
$P(U,u)^{-1}P(U,u')$	$P(u,u')+\gamma(u,u')\nu(U,u)\otimes\nu(u,u')$
$P(u',u)P(U,u)^{-1}P(U,u')$	$P(u')+\delta(U,u,u')\nu(U,u')\otimes\nu(u,u')$
$P(u',u)P(u',U,u)^{-1}$	$P(u')+\delta(U,u,u')\nu(U,u')\otimes[\nu(u,u')-\nu(U,u')]$

where

$$\delta(U,u,u')=\frac{\gamma(U,u')\gamma(u',u)}{\gamma(U,u)},\qquad \delta(U,u,u')^{-1}=\delta(u,U,u')\,, \qquad \text{(A.4.2)}$$

and

$$P(u,U,u')=P(u,U)P(U,u')$$

A.5. Derivatives

Suppose one uses the suggestive notation

$$^{(4)}D(U)/d\tau_U = {}^{(4)}\nabla_U \qquad \text{(A.5.1)}$$

for the "total covariant derivative" along U. Its spatial projection with respect to u and rescaling corresponding to the reparametrization of Eq. (A.3.4) is then given by the "Fermi-Walker total spatial covariant derivative," defined by

$$\begin{aligned} D_{(\text{fw},U,u)}/d\tau_{(U,u)} &= \gamma^{-1}D_{(\text{fw},U,u)}/d\tau_U = \gamma^{-1}P(u)^{(4)}D(U)/d\tau_U \\ &= \nabla_{(\text{fw})}(u)+\nabla(u)_{\nu(U,u)}\,. \end{aligned} \qquad \text{(A.5.2)}$$

Extend this to two other similar derivative operators (the co-rotating Fermi-Walker and the Lie total spatial covariant derivatives) by

$$D_{(\text{tem},U,u)}/d\tau_{(U,u)} = \nabla_{(\text{tem})}(u)+\nabla(u)_{\nu(U,u)}\,, \qquad \text{tem=fw,cfw,lie}\,, \qquad \text{(A.5.3)}$$

which are then related to each other in the same way as the corresponding temporal derivative operators

$$\begin{aligned} D_{(\mathrm{cfw},U,u)}X^\alpha/d\tau_{(U,u)} &= D_{(\mathrm{fw},U,u)}X^\alpha/d\tau_{(U,u)} + \omega(u)^\alpha{}_\beta X^\beta \\ &= D_{(\mathrm{lie},U,u)}X^\alpha/d\tau_{(U,u)} + \theta(u)^\alpha{}_\beta X^\beta \end{aligned} \tag{A.5.4}$$

when acting on a spatial vector field X. All of these derivative operators reduce to the ordinary parameter derivative $D/d\tau_{(U,u)} \equiv d/d\tau_{(U,u)}$ when acting on a function and extend in an obvious way to all tensor fields.

Introduce the ordinary and co-rotating Fermi-Walker and the Lie "relative accelerations" of U with respect to u by

$$a_{(\mathrm{tem})}(U,u) = D_{(\mathrm{tem})}(U,u)\nu(U,u)/d\tau_{(U,u)}\,, \qquad \mathrm{tem{=}fw,cfw,lie}\,. \tag{A.5.5}$$

These are related to each other in the same way as the corresponding derivative operators in Eq. (A.2.15).

The total spatial covariant derivative operators restrict in a natural way to a single timelike worldline with 4-velocity U, where the $D/d\tau$ notation is most appropriate; ${}^{(4)}D(U)/d\tau_U$ is often called the absolute or intrinsic derivative along the worldline of U (associated with an induced connection along such a worldline).

A.6. Applications

A.6.1. Test-particle motion

Let's consider the motion of a unit mass test-particle with four velocity U, accelerated by an external force $f(U)$: $a(U) = f(U)$. A generic observer u can measure the particle four velocity U, obtaining its relative energy $E(U,u) = \gamma(U,u)$ and momentum $p(U,u) = \gamma(U,u)\nu(U,u)$,

$$U = E(U,u)[u + p(U,u)] = \gamma(U,u)[u + \nu(U,u)]. \tag{A.6.1}$$

Splitting the acceleration equation gives the evolution (along U) of the relative energy and momentum of the particle

$$\begin{aligned} \frac{dE(U,u)}{d\tau_{(U,u)}} &= [F^{(G)}_{(\mathrm{tem},U,u)} + F(U,u)]\cdot\nu(U,u) \\ &\quad + \epsilon_{(\mathrm{tem})}\gamma(U,u)\nu(U,u)\cdot(\theta(u)\llcorner\nu(U,u)) \\ \frac{D_{(\mathrm{tem})}p(U,u)}{d\tau_{(U,u)}} &= F^{(G)}_{(\mathrm{tem},U,u)} + F(U,u)\,, \end{aligned} \tag{A.6.2}$$

where tem=fw,cfw,lie,lie$^\flat$ refers to the various possible (i.e. geometrically meaningful) transport of vectors along U, $\epsilon_{(\text{tem})} = (0, 0, -1, 1)$ respectively and

$$\begin{aligned} d\tau_{(U,u)} &= \gamma(U,u) d\tau_U \\ F^{(G)}_{(\text{tem},U,u)} &= \gamma(U,u)[g(u) + H_{(\text{tem},u)} \llcorner \nu(U,u)] \\ F(U,u) &= \gamma(U,u)^{-1} P(u,U) f(U) \end{aligned}$$

with

$$\begin{aligned} H_{(\text{fw},u)} &= \omega(u) - \theta(u) \qquad & H_{(\text{cfw},u)} &= 2\omega(u) - \theta(u) \\ H_{(\text{lie},u)} &= 2\omega(u) - 2\theta(u) \qquad & H_{(\text{lie}^\flat,u)} &= 2\omega(u) \,. \end{aligned} \tag{A.6.3}$$

The gravitoelectric vector field $g(u) = -a(u) = -\nabla_u u$ and the gravitomagnetic vector field $H(u) = 2[^{*(u)}\omega(u)^\flat]^\sharp$ of the observer u (sign-reversed acceleration and twice the vorticity vector field) are defined by the exterior derivative of u

$$du^\flat = [u \wedge g(u) + {}^{*(u)}H(u)]^\flat \,. \tag{A.6.4}$$

and will be essential in showing the analogy between the gravitational force $F^{(G)}_{(\text{tem},U,u)}$ and the Lorentz force. The expansion scalar $\Theta(u) = \operatorname{Tr}\theta(u)$ appears in an additional term in the covariant derivative of u as the trace of the (mixed) expansion tensor $\theta(u)$, of which the shear tensor $\sigma(u) = \theta(u) - \frac{1}{3}\Theta(u)P(u)$ is its tracefree part

$$\nabla u = -a(u) \otimes u^\flat + \theta(u) - \omega(u) \,. \tag{A.6.5}$$

The term $D_{(\text{tem})} p(U,u)/d\tau_{(U,u)}$ contains itself the "spatial geometry" contribution which must be added to the gravitational and the external force to reconstruct the spacetime point of view. Actually, this term comes out naturally and is significant all along the line of the particle: the single terms $\nabla_{(\text{fw},u)}$ and $\nabla(u)_{\nu(U,u)}$, in which it can be further decomposed, are not individually meaningful unless one defines some extension for the spatial momentum $p(U,u)$ off the line of the particle, which of course is unnecessary at all.

From this spatial geometry contribution a general relativistic version of inertial forces can be further extracted.

A.6.2. Maxwell's equations

Maxwell's equations can be expressed covariantly in many ways. For instance, in differential form language one has

$$dF = 0 \,, \qquad d^*F = -4\pi^* J^\flat \,, \tag{A.6.6}$$

where F is the Faraday electromagnetic 2-form and J is the current vector field, obeying the conservation law

$$\delta J^\flat = {}^*d{}^*J^\flat = 0 \,. \tag{A.6.7}$$

The splitting of the electromagnetic 2-form F by any observer family (with unit 4-velocity vector field u) gives the associated electric and magnetic vector fields $E(u)$ and $B(u)$ as measured by those observers through the Lorentz force law on a test charge, and the relative charge and current density $\rho(u)$ and $J(u)$. The "relative observer decomposition" of F and its dual 2-form ${}^*F$ is

$$\begin{aligned} F &= [u \wedge E(u) + {}^{*(u)}B(u)]^\flat \,, \\ {}^*F &= [-u \wedge B(u) + {}^{*(u)}E(u)]^\flat \,, \end{aligned}$$

while J has the representation

$$J = \rho(u)u + J(u) \,. \tag{A.6.8}$$

If U is the 4-velocity of any test particle with charge q and nonzero rest mass m, it has the orthogonal decomposition

$$U = \gamma(U,u)[u + \nu(U,u)] \,. \tag{A.6.9}$$

Its absolute derivative with respect to a proper time parametrization of its world line is its 4-acceleration $a(U) = DU/d\tau_U$. The Lorentz force law then takes the form

$$ma(U) = q\gamma(U,u)[E(u) + \nu(U,u) \times_u B(u)] \,. \tag{A.6.10}$$

The relative observer formulation of Maxwell's equations is well known. Projection of the differential form equations (A.6.6) along and orthogonal to u gives the spatial scalar (divergence) and spatial vector (curl) equations:

$$\begin{aligned} &\mathrm{div}_u B(u) + \vec{H}(u) \cdot_u E(u) = 0 \,, \\ &\mathrm{curl}_u E(u) - \vec{g}(u) \times_u E(u) + [\pounds(u)_u + \Theta(u)]B(u) = 0 \,, \\ &\mathrm{div}_u E(u) - \vec{H}(u) \cdot_u B(u) = 4\pi\rho(u) \,, \\ &\mathrm{curl}_u B(u) - \vec{g}(u) \times_u B(u) - [\pounds(u)_u + \Theta(u)]E(u) = 4\pi J(u) \,, \end{aligned} \tag{A.6.11}$$

This representation of Maxwell's equations differs from the Ellis representation only in the use of the spatially projected Lie derivative rather than the spatially projected covariant derivative along u (spatial Fermi-Walker derivative). These two derivative operators are related by the following identity for a spatial vector field X (orthogonal to u)

$$[\pounds(u)_u + \Theta(u)]X = [\nabla(u)_u + \{-\sigma(u) + \omega(u)\} \llcorner]X\,. \tag{A.6.12}$$

It is clear, at this point, that for any spacetime tensor equation the "1+3" associated version allows one to read it in a Newtonian form and to interpret it quasi-classically.

For instance one can consider motion of test fields in a given gravitational background (i.e. neglecting backreaction) as described by spacetime equations and look at their "1+3" counterpart. Over the last ten years, in a similar way in which we have discussed the splitting of Maxwell's equations in integral formulation, we have studied scalar field, spinorial field (Dirac fields), fluid motions, etc.

B. Motion of particles and extended bodies in General Relativity

B.1. Introduction

The motion of an extended body in a given background may be studied by treating the body via a multipole expansion. The starting point of this method is the covariant conservation law

$$\nabla_\mu T^\mu{}_\nu = 0 \,, \tag{B.1.1}$$

where $T^{\mu\nu}$ is the energy-momentum tensor describing the body. The body sweeps out a narrow tube in spacetime as it moves. Let L be a line inside this tube representing the motion of the body. Denote the coordinates of the points of this line by X^α, and define the displacement $\delta x^\alpha = X^\alpha - x^\alpha$, where x^α are the coordinates of the points of the body. Let us consider now the quantities

$$\int T^{\mu\nu} \mathrm{d}V \,, \qquad \int \delta x^\lambda T^{\mu\nu} \mathrm{d}V \,, \qquad \int \delta x^\lambda \delta x^\rho T^{\mu\nu} \mathrm{d}V \,, \quad \ldots \tag{B.1.2}$$

where the integrations are carried out on the 3-dimensional hypersurfaces of fixed time $t = X^0 = const$, the tensor $T^{\mu\nu}$ being different from zero only inside the world tube: these are the successive terms of the multipole expansion. A single-pole particle is defined as a particle that has nonvanishing at least some of the integrals in the first (monopole) term, assuming that all the integrals containing δx^μ vanish. A pole-dipole particle, instead, is defined as a particle for which all the integrals with more than one factor of δx^μ (dipole term) vanish. Higher order approximations may be defined in a similar way. Thus, a single-pole particle is a test particle without any internal structure. A pole-dipole particle, instead, is a test particle whose internal structure is expressed by its spin, an antisymmetric second-rank tensor defined by

$$S^{\mu\nu} \equiv \int \left[\delta x^\mu T^{0\nu} - \delta x^\nu T^{0\mu} \right] \mathrm{d}V \,. \tag{B.1.3}$$

The equations of motion are, then, obtained by applying the Einstein's field equations together with conservation of the energy-momentum tensor (B.1.1) describing the body. For a single-pole particle this leads to a free particle moving along the geodesics associated with the given background field. For the motion of a pole-dipole particle, instead, the corresponding set of equations was derived by Papapetrou [84] by using the above procedure. Obviously, the model is worked out under the assumption that the dimensions of the test particle are very small compared with the characteristic length of the basic field (i.e., with backreaction neglected), and that the gravitational radiation emitted by the particle in its motion is negligible. As a final remark, note that this model can be extended to charged bodies by considering in addition the conservation law of the current density.

B.2. The Mathisson-Papapetrou model

The equations of motion for a spinning (or pole-dipole) test particle in a given gravitational background were deduced by Mathisson and Papapetrou [84, 85] and read

$$\frac{DP^\mu}{\mathrm{d}\tau_U} = -\frac{1}{2}R^\mu{}_{\nu\alpha\beta}U^\nu S^{\alpha\beta} \equiv F^{(\mathrm{spin})\mu} \,, \tag{B.2.1}$$

$$\frac{DS^{\mu\nu}}{\mathrm{d}\tau_U} = P^\mu U^\nu - P^\nu U^\mu \,, \tag{B.2.2}$$

where P^μ is the total four-momentum of the particle, and $S^{\mu\nu}$ is a (antisymmetric) spin tensor; U is the timelike unit tangent vector of the "center of mass line" used to make the multipole reduction. Equations (B.2.1) and (B.2.2) define the evolution of P and S only along the world line of U, so a correct interpretation of U is that of being tangent to the *true* world-line of the spinning particle. The 4-momentum P and the spin tensor S are then defined as vector fields along the trajectory of U. By contracting both sides of Eq. (B.2.2) with U_ν, one obtains the following expression for the total 4-momentum

$$P^\mu = -(U\cdot P)U^\mu - U_\nu \frac{DS^{\mu\nu}}{\mathrm{d}\tau_U} \equiv mU^\mu + P_s^\mu \,, \tag{B.2.3}$$

where $m = -U\cdot P$ reduces to the ordinary mass in the case in which the particle is not spinning, and P_s is a 4-vector orthogonal to U.

The test character of the particle under consideration refers to its mass as well as to its spin, since both quantities should not be large enough to contribute to the background metric. In what follows, with the magnitude of the spin of the particle, with the mass and with a natural lengthscale associated with the gravitational background we will construct a dimensionless parameter as a smallness indicator, which we retain to the first order only so that the test character of the particle be fully satisfied. Moreover, in order to have a closed set of equations Eqs. (B.2.1) and (B.2.2) must be completed with supplementary conditions (SC), whose standard choices in the literature are the following

1. Corinaldesi-Papapetrou [86] conditions (CP): $S^{\mu\nu}(e_0)_\nu = 0$, where e_0 is the coordinate timelike direction given by the background;

2. Pirani [87] conditions (P): $S^{\mu\nu}U_\nu = 0$;

3. Tulczyjew [88] conditions (T): $S^{\mu\nu}P_\nu = 0$;

all of these are algebraic conditions.

Detailed studies concerning spinning test particles in General Relativity are due to Dixon [89, 90, 91, 92, 93], Taub [94], Mashhoon [95, 96] and Ehlers and Rudolph [97]. The Mathisson-Papapetrou model does not give *a priori* restrictions on the causal character of U and P and there is no agreement in the literature on how this point should be considered. For instance, Tod, de Felice and Calvani [98] consider P timelike, assuming that it represents the total energy momentum content of the particle, while they do not impose any causality condition on the world line U, which plays the role of a mere mathematical "tool" to perform the multipole reduction. Differently, according to Mashhoon [96], P can be considered analogously to the canonical momentum of the particle: hence, there should be not any meaning for its causality character, while the world line U has to be timelike (or eventually null) because it represents the center of mass line of the particle. This uncertainty in the model itself then reflects in the need for a supplementary condition, whose choice among the three mentioned above is arbitrary, making the general relativistic description of a spinning test particle somehow unsatisfactory. When both U and P are timelike vectors as e_0, all of them can be taken as the 4-velocity field of a preferred observer family, and all the SC above state that for the corresponding observer the spin tensor is purely spatial. In a sense, only P and T supplementary conditions give "intrinsic" relations between the various unknown of the model and they should be somehow more physical conditions. In fact, the CP conditions are "coordinate dependent," being e_0 the coordinate timelike vector. It is worth to mention that grounded on physical reasons, Dixon has shown that the T conditions should be preferred with respect to the others.

B.3. The Dixon-Souriau model

The equations of motion for a charged spinning test particle in a given gravitational as well as electromagnetic background were deduced by Dixon-

Souriau [99, 100, 101, 102]. They have the form

$$\frac{DP^\mu}{\mathrm{d}\tau_U} = -\frac{1}{2}R^\mu{}_{\nu\alpha\beta}U^\nu S^{\alpha\beta} + qF^\mu{}_\nu U^\nu - \frac{\lambda}{2}S^{\rho\sigma}\nabla^\mu F_{\rho\sigma} \equiv F^{(\mathrm{tot})\mu} , \quad \text{(B.3.1)}$$

$$\frac{DS^{\mu\nu}}{\mathrm{d}\tau_U} = P^\mu U^\nu - P^\nu U^\mu + \lambda[S^{\mu\rho}F_\rho{}^\nu - S^{\nu\rho}F_\rho{}^\mu] , \quad \text{(B.3.2)}$$

where $F^{\mu\nu}$ is the electromagnetic field, P^μ is the total 4-momentum of the particle, and $S^{\mu\nu}$ is the spin tensor (antisymmetric); U is the timelike unit tangent vector of the "center of mass line" used to make the multipole reduction. As it has been shown by Souriau, the quantity λ is an arbitrary electromagnetic coupling scalar constant. We note that the special choice $\lambda = -q/m$ (see [46]) in flat spacetime corresponds to the Bargman-Michel-Telegdi [103] spin precession law.

B.4. Particles with quadrupole structure

The equations of motion for an extended body in a given gravitational background were deduced by Dixon [89, 90, 91, 92, 93] in multipole approximation to any order. In the quadrupole approximation they read

$$\frac{DP^\mu}{\mathrm{d}\tau_U} = -\frac{1}{2}R^\mu{}_{\nu\alpha\beta}U^\nu S^{\alpha\beta} - \frac{1}{6}J^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}{}^{;\mu} \equiv F^{(\mathrm{spin})\mu} + F^{(\mathrm{quad})\mu} \quad \text{(B.4.1)}$$

$$\frac{DS^{\mu\nu}}{\mathrm{d}\tau_U} = 2P^{[\mu}U^{\nu]} - \frac{4}{3}J^{\alpha\beta\gamma[\mu}R^{\nu]}{}_{\alpha\beta\gamma} , \quad \text{(B.4.2)}$$

where $P^\mu = mU_p^\mu$ (with $U_p \cdot U_p = -1$) is the total four-momentum of the particle, and $S^{\mu\nu}$ is a (antisymmetric) spin tensor; U is the timelike unit tangent vector of the "center of mass line" $\mathcal{C}_U$ used to make the multipole reduction, parametrized by the proper time τ_U. The tensor $J^{\alpha\beta\gamma\delta}$ is the quadrupole moment of the stress-energy tensor of the body, and has the same algebraic symmetries as the Riemann tensor. Using standard spacetime splitting techniques it can be reduced to the following form

$$J^{\alpha\beta\gamma\delta} = \Pi^{\alpha\beta\gamma\delta} - \bar{u}^{[\alpha}\pi^{\beta]\gamma\delta} - \bar{u}^{[\gamma}\pi^{\delta]\alpha\beta} - 3\bar{u}^{[\alpha}Q^{\beta][\gamma}\bar{u}^{\delta]} , \quad \text{(B.4.3)}$$

where $Q^{\alpha\beta} = Q^{(\alpha\beta)}$ represents the quadrupole moment of the mass distribution as measured by an observer with 4-velocity $\bar{u}$. Similarly $\pi^{\alpha\beta\gamma} = \pi^{\alpha[\beta\gamma]}$ (with the additional property $\pi^{[\alpha\beta\gamma]} = 0$) and $\Pi^{\alpha\beta\gamma\delta} = \Pi^{[\alpha\beta][\gamma\delta]}$ are essentially the body's momentum and stress quadrupoles. Moreover the various fields $Q^{\alpha\beta}$, $\pi^{\alpha\beta\gamma}$ and $\Pi^{\alpha\beta\gamma\delta}$ are all spatial (i.e. give zero after any contraction by $\bar{u}$). The number of independent components of $J^{\alpha\beta\gamma\delta}$ is 20: 6 in $Q^{\alpha\beta}$, 6 in $\Pi^{\alpha\beta\gamma\delta}$ and 8 in $\pi^{\alpha\beta\gamma}$. When the observer $\bar{u} = U_p$, i.e. in the frame associated with the momentum of the particle, the tensors $Q^{\alpha\beta}$, $\pi^{\alpha\beta\gamma}$ and $\Pi^{\alpha\beta\gamma\delta}$ have an intrinsic meaning.

There are no evolution equations for the quadrupole as well as higher multipoles as a consequence of the Dixon's construction, so their evolution is completely free, depending only on the considered body. Therefore the system of equations is not self-consistent, and one must assume that all unspecified quantities are known as intrinsic properties of the matter under consideration.

In order the model to be mathematically correct the following additional condition should be imposed to the spin tensor:

$$S^{\mu\nu}U_{p\nu} = 0. \tag{B.4.4}$$

Such supplementary conditions (or Tulczyjew-Dixon conditions [88, 89]) are necessary to ensure the correct definition of the various multipolar terms.

Dixon's model for structured particles originated to complete and give a rigorous mathematical support to the previously introduced Mathisson-Papapetrou model [84, 85, 86, 87], i.e. a multipole approximation to any order which includes evolutional equations along the "center line" for all the various structural quantities. The models are then different and a comparison between the two is possible at the dipolar order but not once the involved order is the quadrupole.

Here we limit our considerations to Dixon's model under the further simplifying assumption[94, 97] that the only contribution to the complete quadrupole moment $J^{\alpha\beta\gamma\delta}$ stems from the mass quadrupole moment $Q^{\alpha\beta}$, so that $\pi^{\alpha\beta\gamma} = 0 = \Pi^{\alpha\beta\gamma\delta}$ and

$$J^{\alpha\beta\gamma\delta} = -3U_p^{[\alpha}Q^{\beta][\gamma}U_p^{\delta]}\,, \qquad Q^{\alpha\beta}U_{p\beta} = 0\,; \tag{B.4.5}$$

The assumption that the particle under consideration is a test particle means that its mass, its spin as well as its quadrupole moments must all be small enough not to contribute significantly to the background metric. Otherwise, backreaction must be taken into account.

B.5. Null multipole reduction world line: the massless case

The extension of the Mathisson-Papapetrou model to the case of a null multipole reduction world line l has been considered by Mashhoon [96]: the model equations have exactly the same form as (B.2.1) and (B.2.2), with U (timelike) replaced by l (null) for what concerns the multipole reduction world line and τ_U (proper time parametrization of the U line) replaced by λ (affine parameter along the l line):

$$\frac{DP^\alpha}{d\lambda} = -\frac{1}{2}R^\alpha{}_{\beta\rho\sigma}l^\beta S^{\rho\sigma} \equiv F^{(\text{spin})\alpha}, \qquad \text{(B.5.1)}$$

$$\frac{DS^{\alpha\beta}}{d\lambda} = [P \wedge l]^{\alpha\beta}. \qquad \text{(B.5.2)}$$

Equations (B.5.1) and (B.5.2) should be then solved assuming some SC. Let us limit ourselves to the case of "intrinsic" SC, i.e. Pirani and Tulczyjew, with Pirani's conditions now naturally generalized as $S^{\alpha\beta}l_\beta = 0$. Furthermore, we require $P \cdot l = 0$: in fact, we are interested to the massless limit of the Mathisson-Papapetrou equations, and as the mass of the particle is defined by $m = -P \cdot U$ the massless limit implies $-P \cdot l = 0$.

By denoting with $\{l = e_1, n = e_2, m = e_3, \bar{m} = e_4\}$ a complex null frame along the center line l, such that $l \cdot l = n \cdot n = m \cdot m = 0$, $l \cdot n = 1$, $l \cdot m = l \cdot m = 0$ and $m \cdot \bar{m} = -1$, it is possible to parametrize the path so that

$$\frac{Dl^\mu}{d\lambda} = \bar{b}m^\mu + b\bar{m}^\mu,$$

$$\frac{Dn^\mu}{d\lambda} = \bar{a}m^\mu + a\bar{m}^\mu,$$

$$\frac{Dm^\mu}{d\lambda} = al^\mu + bn^\mu + icm^\mu, \qquad \text{(B.5.3)}$$

where a, b, c are functions of λ and c is real. The metric signature is assumed now $+---$ in order to follow standard notation of Newman-Penrose formalism, and the bar over a quantity denotes complex conjugation. Equations (B.5.3) are the analogous of the FS relations for null lines so that, repeating exactly the above procedure, one gets the final set of equations. Since for a massless spinning test particle we have $m = -P \cdot l = 0$, the total 4-momentum P has the following decomposition:

$$P^\mu = -[Bl^\mu + Am^\mu + \bar{A}\bar{m}^\mu]. \qquad \text{(B.5.4)}$$

Following Mashhoon [96], Tulczyjew's conditions $S^{\alpha\beta}P_\beta = 0$ are in general inconsistent in the presence of a gravitational background if in addition one has P lightlike: $P \cdot P = 0$. Thus, even if these inconsistencies concern only the case of null P, we are clearly forced to consider Pirani's SC as the only physically meaningful supplementary conditions. Using the P supplementary conditions (implying $b = 0$), Mashhoon has shown that l is necessarily geodesics: $Dl^\mu/d\lambda = 0$ and

$$S^{\mu\nu} = f(\lambda)[l \wedge m]^{\mu\nu} + \bar{f}(\lambda)[l \wedge \bar{m}]^{\mu\nu} + ig(\lambda)[m \wedge \bar{m}]^{\mu\nu}, \qquad \text{(B.5.5)}$$

with B real and

$$A = \frac{df}{d\lambda} + icf - ig\bar{a}\,, \qquad P^\mu P_\mu = -2|A|^2\,. \tag{B.5.6}$$

so that P is in general spacelike or eventually null. Furthermore, he has shown that the spin vector defined by

$$\mathcal{S}^\mu = \frac{1}{2}\eta^{\mu\nu\alpha\beta} l_\nu S_{\alpha\beta} \tag{B.5.7}$$

is constant along l and either parallel or antiparallel to l.

Finally, the generalized momentum of the particle should be determined by solving equations (B.5.1) and (B.5.2) supplemented by $S^{\alpha\beta}l_\beta = 0$. The other components of the spin tensor not summarized by the spin vector should be determined too. By assuming $a = 0$ (n parallel propagated along l) without any loss of the physical content of the solution, Mashhoon has obtained for f and B the following differential equations:

$$\begin{aligned}
\left[\frac{d}{d\lambda} + ic\right]^2 f &= fR_{1413} + \bar{f}R_{1414} + igR_{1434}\,, \\
-\frac{dB}{d\lambda} &= fR_{1213} + \bar{f}R_{1214} + igR_{1234}\,,
\end{aligned} \tag{B.5.8}$$

which determine the total 4-momentum and the spin tensor along the path once they have been specified initially.

B.6. Applications

B.6.1. The special case of constant frame components of the spin tensor

Due to the mathematical complexity in treating the general case of non-constant frame components of the spin tensor, we have considered first the simplest case of massive spinning test particles moving uniformly along circular orbits with constant frame components of the spin tensor with respect to a naturally geometrically defined frame adapted to the stationary observers in the Schwarzschild spacetime [104] as well as in other spacetimes of astrophysical interest: Reissner-Nordström spacetime [105], Kerr spacetime [5], superposed static Weyl field [106], vacuum C metric [107]. A static spin vector is a very strong restriction on the solutions of the Mathisson-Papapetrou equations of motion. However, this assumption not only greatly simplifies the calculation, but seems to be not so restrictive, since, as previously demonstrated at least in the Schwarzschild case, the spin tensor components still remain constant under the CP an T choices of supplementary conditions, starting from the more general non-constant case.

We have confined our attention to spatially circular equatorial orbits in Schwarzschild, Reissner-Nordström and Kerr spacetimes, and searched for observable effects which could eventually discriminate among the standard supplementary conditions. We have found that if the world line chosen for the multipole reduction and whose unit tangent we denote as U is a circular orbit, then also the generalized momentum P of the spinning test particle is tangent to a circular orbit even though P and U are not parallel 4-vectors. These orbits are shown to exist because the spin induced tidal forces provide the required acceleration no matter what supplementary condition we select. Of course, in the limit of a small spin the particle's orbit is close of being a circular geodesic and the (small) deviation of the angular velocities from the geodesic values can be of an arbitrary sign, corresponding to the possible spin-up and spin-down alignment to the z-axis. When two massive particles (as well as photons) orbit around a gravitating source in opposite directions, they make one loop with respect to a given static observer with different arrival times. This difference is termed clock effect (see [50, 108, 109, 110, 111] and references therein). Hereafter we shall refer the co/counter-rotation as with respect to a fixed sense of variation of the azimuthal angular coordinate. In the case of a static observer and of timelike spatially circular geodesics the coordinate time delay is given by

$$\Delta t_{(+,-)} = 2\pi \left(\frac{1}{\zeta_+} + \frac{1}{\zeta_-} \right) , \tag{B.6.1}$$

where $\zeta_\pm$ denote angular velocities of two opposite rotating geodesics. In the case of spinless neutral particles in geodesic motion on the equatorial plane of both Schwarzschild and Reissner-Nordström spacetimes one has $\zeta_+ = -\zeta_-$, and so the clock effect vanishes; in the Kerr case, instead, the clock effect reads $\Delta t_{(+,-)} = 4\pi a$, where a is the angular momentum per unit mass of the Kerr black hole. These results are well known in the literature. We have then extended the notion of clock effect to non geodesic circular trajectories considering co/counter-rotating spinning-up/spinning-down particles. In this case we have found that the time delay is nonzero for oppositely orbiting both spin-up or spin-down particles even in both Schwarzschild and Reissner-Nordström cases, and can be measured. In addition, we have found that a nonzero gravitomagnetic clock effect appears in the Reissner-Nordström spacetime for spinless (oppositely) charged particles as well.

An analogous effect is found in the case of superposed Weyl fields corresponding to Chazy-Curzon particles and Schwarzschild black holes when the circular motion of spinning test particles is considered on particular symmetry hyperplanes, where the orbits are close to a geodesic for small values of the spin. In the case of the C metric, instead, we have found that the orbital frequency is in general spin-dependent, but there is no clock effect, in contrast to the limiting Schwarzschild case.

B.6.2. Spin precession in Schwarzschild and Kerr spacetimes

We have then studied the behaviour of spinning test particles moving along equatorial circular orbits in the Schwarzschild [6] as well as Kerr [7] spacetimes within the framework of the Mathisson-Papapetrou approach supplemented by standard conditions, in the general case in which the components of the spin tensor are not constant along the orbit. We have found that precession effects occur only if the Pirani's supplementary conditions are imposed, where one finds a Fermi-Walker transported spin vector along an accelerated center of mass world line. The remaining two supplementary conditions apparently force the test particle center of mass world line to deviate from a circular orbit because of the feedback of the precessing spin vector; in addition, under these choices of supplementary conditions the spin tensor components still remain constant. In reaching these conclusions, we only considered solutions for which both U and P are timelike vectors, in order to have a meaningful interpretation describing a spinning test particle with nonzero rest mass.

B.6.3. Massless spinning test particles in vacuum algebraically special spacetimes

As a final application, we have derived the equations of motion for massless spinning test particles in general vacuum algebraically special spacetimes, using the Newman-Penrose formalism, in the special case in which the multipole reduction world line is aligned with a principal null direction of the spacetime [112]. This situation gives very simple equations and their complete integration is straightforward. Explicit solutions corresponding to some familiar Petrov type D and type N spacetimes (including Schwarzschild, Taub-NUT, Kerr, C metric, Kasner, single exact gravitational wave) are derived and discussed. Furthermore, we have investigated the motion along (null) circular orbits, providing explicit solutions in black hole spacetimes.

B.6.4. Quadrupole effects in black hole spacetimes

We have studied the motion of quadrupolar particles on a Schwarzschild as well as Kerr backgrounds [113, 114] following Dixon's model. In the simplified situation of constant frame components (with respect to a natural orthonormal frame) of both the spin and the quadrupole tensor of the particle we have found the kinematical conditions to be imposed to the particle's structure in order the orbit of the particle itself be circular and confined on the equatorial plane. Co-rotating and counter-rotating particles result to have a non-symmetric speed in spite of the spherical symmetry of the background, due to their internal structure. This fact has been anticipated when studying spinning particles only, i.e. with vanishing quadrupole moments. We show modifications due to the quadrupole which could be eventually observed in experiments. Such experiment, however, cannot concern standard clock effects, because in this case we have shown that there are no contributions arising from the quadrupolar structure of the body. In contrast, the effect of the quadrupole terms could be important when considering the period of revolution of an extended body around the central source: measuring the period will provide an estimate of the quantities determining the quadrupolar structure of the body, if its spin is known.

It would be of great interest to extend this analysis to systems with varying quadrupolar structure and emitting gravitational waves without perturbing significantly the background spacetime.

B.6.5. Quadrupole effects in gravitational wave spacetimes

We have studied how a small extended body at rest interacts with an incoming single plane gravitational wave. The body is spinning and also endowed with a quadrupolar structure, so that due to the latter property it can be thus considered as a good model for a gravitational wave antenna.

We have first discussed the motion of such an extended body by assuming that it can be described according to Dixon's model and that the gravitational field of the wave is weak, i.e. the "reaction" (induced motion) of a "gravitational wave antenna" (the extended body) to the passage of the wave, and then the case of an exact plane gravitational wave. We have found that in general, even if initially absent, the body acquires a dipolar moment induced by the quadrupole tensor, a property never pointed out before in the literature.

Special situations may occur in which certain spin components change their magnitude leading to effects (e.g. spin-flip) which can be eventually observed. This interesting feature recalls the phenomenon of glitches observed in pulsars: a sudden increase in the rotation frequency, often accompanied by an increase in slow-down rate. The physical mechanism triggering glitches is not well understood yet, even if these are commonly thought to be caused by internal processes. If one models a pulsar by a Dixon's extended body, then the present analysis shows that a sort of glitch can be generated by the passage of a strong gravitational wave, due to the pulsar quadrupole structure. In fact, we have found that the profile of a polarization function can be suitably selected in order to fit observed glitches and in particular to describe the post-glitch behavior.

B.6.6. Quadrupolar particles and the equivalence principle

We have compared the two “reciprocal ” situations of motion of an extended body endowed with structure up to the mass quadrupole moment in a Shwarzschild background spacetime (as described by Dixon’s model) with that of a test particle in geodesic motion in the background of an exact solution of Einstein’s field equations describing a source with quadrupolar structure (for a more detailed study of this kind of solutions generalizing Schwarzschild, Kerr and Kerr-Newman spacetimes see also the section “Generalizations of the Kerr-Newman solution,” included in the present report). Under certain conditions the two situations give perfect corresponding results a fact which has been interpreted as an argument in favour of the validity of Dixon’s model.

B.6.7. Poynting-Robertson-like effects

Test particle motion in realistic gravitational fields is of obvious astrophysical importance and at the same time it provides reliable evidence of the properties of those gravitational fields. However, in many actual astrophysical systems the particles are not moving freely but are influenced by ambient matter, electromagnetic fields and radiation. In typical situations, these “physical” effects are probably even more important than fine details of the spacetime geometry alone. The most remarkable conditions, from the point of view of general relativity as well as astrophysics, appear near very compact objects where both the pure gravitational and other “physical” effects typically become extraordinarily strong.

In a series of papers, recently, we have focused on the motion of test particles in a spherically symmetric gravitational field, under the action of a Thomson-type interaction with radiation emitted or accreted by a compact center. This kind of problem was first investigated by Poynting using Newtonian gravity and then in the framework of linearized general relativity by Robertson (see [165] and the references therein). It involves competition between gravity and radiation drag, which may lead to interesting types of motion which do not occur in strictly vacuum circumstances. In particular, there arises the question of whether equilibrium behavior like circular orbit motion or even “staying at rest” are possible in some cases. Theoretical aspects of the Poynting-Robertson effect as well as its astrophysical relevance in specific situations have been studied by many authors since the original pioneering work. We first considered this effect on test particles orbiting in the equatorial plane of a Schwarzschild or Kerr black hole, assuming that the source of radiation is located symmetrically not far from the horizon (in the case of outgoing flux). Successively, we have generalized these results by including in our discussion other relevant spacetimes, e.g. Vaidya, or considering spinning particles undergoing Poynting-Robertson-like effect.

C. Metric and curvature perturbations in black hole spacetimes

C.1. Perturbations of charged and rotating Black hole

The gravitational and electromagnetic perturbations of the Kerr-Newman metric represent still an open problem in General Relativity whose solution could have an enormous importance for the astrophysics of charged and rotating collapsed objects. A complete discussion about this problems needs a plenty of different mathematical tools: the Newman-Penrose formalism in the tetradic and spinor version, the Cahen-Debever-Defrise self dual theory, the properties of the spin-weighted angular harmonics, with particular attention to the related differential geometry and the group theory, some tools of complex analysis, etc, but in any case it is difficult to handle with the perturbative equations. Fortunately, during the last years, the modern computers and software have reached an optimal computational level which allows now to approach this problem from a completely new point of view.

The Kerr-Newman solution in Boyer-Lindquist coordinates is represented by the metric:

$$\begin{aligned} ds^2 &= \left(1-\frac{V}{\Sigma}\right)dt^2 + \frac{2a\sin^2\theta}{\Sigma}V dt d\phi - \frac{\Sigma}{\Delta}dr^2 \\ &\quad -\Sigma d\theta^2 - \left[r^2+a^2+\frac{a^2\sin^2\theta}{\Sigma}V\right]\sin^2\theta d\phi^2 \end{aligned} \tag{C.1.1}$$

where as usual:

$$\begin{aligned} V &\equiv 2Mr - Q^2 \\ \Delta &\equiv r^2 - 2Mr + a^2 + Q^2 \\ \Sigma &\equiv r^2 + a^2\cos^2\theta \end{aligned} \tag{C.1.2}$$

and by the vector potential:

$$A^\flat = A_\mu \, dx^\mu = \frac{Qr}{\Sigma}(dt - a\sin^2\theta \, d\phi) \ . \tag{C.1.3}$$

To investigate the geometrical features of this metric it is convenient to introduce a symmetry-adapted tetrad. For any type D metric, and in particular for the Kerr-Newman solution, the best choice is a null tetrad with two "legs" aligned along the two repeated principal null directions of the Weyl tensor. The standard theory for analyzing different spin massless wave fields in a given background is represented by the spinorial tetradic formalism of Newman-Penrose (hereafter N-P)[115]. Here we follow the standard approach, pointing out that a more advanced reformulation of this formalism, called "GHP" [116] exists, allowing a more geometric comprehension of the theory. In the N-P formalism, this solution is represented by the following quantities [117] (in this section we use an A label over all quantities for a reason which will be clear later). The Kinnersley tetrad [118]:

$$\begin{aligned}
(l^\mu)^A &= \frac{1}{\Delta}[r^2 + a^2, \Delta, 0, a] \\
(n^\mu)^A &= \frac{1}{2\Sigma}[r^2 + a^2, -\Delta, 0, a] \\
(m^\mu)^A &= \frac{1}{\sqrt{2}(r + ia\cos\theta)}\,[ia\sin\theta, 0, 1, \frac{i}{\sin\theta}] \ ,
\end{aligned} \tag{C.1.4}$$

with the 4^{th} leg represented by the conjugate $(m^{*\mu})^A$, gives the metric tensor of Kerr-Newman spacetime the form:

$$\eta_{(a)(b)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} . \tag{C.1.5}$$

The Weyl tensor is represented by:

$$\begin{aligned}
\Psi_0^A = \Psi_1^A = \Psi_3^A = \Psi_4^A = 0 \\
\Psi_2^A = M\rho^3 + Q^2\rho^*\rho^3
\end{aligned} \tag{C.1.6}$$

and the electromagnetic field is given by:

$$\phi_0^A = \phi_2^A = 0 \ , \ \phi_1^A = \frac{Q}{2(r - ia\cos\theta)^2} \ . \tag{C.1.7}$$

For the Ricci tensor and the curvature scalar we have:

$$\Lambda^A = 0 \quad , \quad \Phi^A_{nm} = 2\phi^A_m \phi^{*A}_n \qquad (\mathrm{m},\mathrm{n} = 0,1,2) \tag{C.1.8}$$

so in Kerr-Newman, the only quantity different from zero is:

$$\Phi^A_{11} = \frac{Q^2}{2\Sigma^2} \; . \tag{C.1.9}$$

The spin coefficients, which are linear combination of the Ricci rotation coefficients, are given by:

$$\begin{aligned}
&\kappa^A = \sigma^A = \lambda^A = \nu^A = \epsilon^A = 0 \; , \\
&\rho^A = \frac{-1}{(r - ia\cos\theta)} \quad , \quad \tau^A = \frac{-ia\rho^A\rho^{*A}\sin\theta}{\sqrt{2}} \; , \\
&\beta^A = \frac{-\rho^{*A}\cot\theta}{2\sqrt{2}} \quad , \quad \pi^A = \frac{ia(\rho^A)^2\sin\theta}{\sqrt{2}} \; , \\
&\mu^A = \frac{(\rho^A)^2\rho^{*A}\Delta}{2} \quad , \; \gamma^A = \mu^A + \frac{\rho^A\rho^{*A}(r-M)}{2} \; , \\
&\alpha^A = \pi^A - \beta^{*A} \; .
\end{aligned} \tag{C.1.10}$$

The directional derivatives are expressed by:

$$D = l^\mu\partial_\mu \, , \quad \Delta = n^\mu\partial_\mu \, , \quad \delta = m^\mu\partial_\mu \, , \quad \delta^* = m^{*\mu}\partial_\mu \, . \tag{C.1.11}$$

Unfortunately in the literature the same letter for (C.1.2) and for the directional derivative along **n** it is used. However the meaning of Δ will always be clear from the context. The study of perturbations in the N-P formalism is achieved splitting all the relevant quantities in the form $l = l^A + l^B, \Psi_4 = \Psi_4^A + \Psi_4^B$, $\sigma = \sigma^A + \sigma^B, D = D^A + D^B$, etc., where the A terms are the background and the B's are small perturbations. The full set of perturbative equations is obtained inserting these quantities in the basic equations of the theory (Ricci and Bianchi identities, Maxwell, Dirac, Rarita-Schwinger equations etc.) and keeping only first order terms. After certain standard algebraic manipulations one usually obtains coupled linear PDE's involving curvature quantities. In the following, we will omit the A superscript for the background quantities. Comparing with the standard Regge-Wheeler-Zerilli [119, 120] approach which gives the equation for the metric, here one gets the equations for Weyl tensor components. This theory is known as *curvature perturbations*. In the case of Einstein-Maxwell perturbed metrics, one gets as in R-W-Z the well known phenomenon of the "gravitationally induced electromagnetic radiation and vice versa" [121], which couples gravitational and electromagnetic fields. In the first formulation, one gets a coupled system for $F^B_{\mu\nu}$ and $g^B_{\mu\nu}$ quantities. In the N-P approach one has the coupling between perturbed Weyl and Maxwell tensor components, although it's possible to recover the metric perturbations using the curvature one [122]. A discussion about the connections between these two approaches can be found in [123]. To make a long story short, taking in account the two Killing vectors of this spacetime, one can write the unknown functions in the form:

$$F(t,r,\theta,\phi) = e^{-i\omega t}e^{im\phi}f(r,\theta) \ . \tag{C.1.12}$$

In the easier cases of Kerr, Reissner-Nordstrom and Schwarzchild, writing $f(r,\theta) = R(r)Y(\theta)$ one gets separability of the problem. For instance, the Reissner-Nordström case [124] is separable using the spin-weighted spherical harmonics:

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) - \left(\frac{m^2+s^2+2ms\cos\theta}{\sin^2\theta}\right)\right]{}_sY^m{}_l(\theta) = -l(l+1){}_sY^m{}_l(\theta) \tag{C.1.13}$$

and their related laddering operators:

$$\left(\frac{d}{d\theta} - \frac{m}{\sin\theta} - s\frac{\cos\theta}{\sin\theta}\right){}_sY^m{}_l(\theta) = -\sqrt{(l-s)(l+s+1)}{}_{s+1}Y^m{}_l(\theta) \tag{C.1.14}$$

$$\left(\frac{d}{d\theta} + \frac{m}{\sin\theta} + s\frac{\cos\theta}{\sin\theta}\right){}_sY^m{}_l(\theta) = +\sqrt{(l+s)(l-s+1)}{}_{s-1}Y^m{}_l(\theta) \ . \tag{C.1.15}$$

The unknown functions can be cast in the form:

$$\begin{aligned}
\Psi_0^B &= e^{-i\omega t}e^{im\phi}{}_2Y^m{}_l(\theta)R_l^{(2)}(r) \\
\chi_1^B &= e^{-i\omega t}e^{im\phi}{}_1Y^m{}_l(\theta)R_l^{(1)}(r) \\
\chi_{-1}^B &= e^{-i\omega t}e^{im\phi}{}_{-1}Y^m{}_l(\theta)\frac{\Delta}{2r^2}R_l^{(-1)}(r) \\
\Psi_4^B &= e^{-i\omega t}e^{im\phi}{}_{-2}Y^m{}_l(\theta)\frac{\Delta^2}{4r^4}R_l^{(-2)}(r)
\end{aligned} \tag{C.1.16}$$

where $\Delta = r^2 - 2Mr + Q^2$, and after manipulations, one gets two sets of coupled ODE's. The first set is:

$$\Big[-\omega^2\frac{r^4}{\Delta}+4i\omega r\left(-2+\frac{r(r-M)}{\Delta}+\frac{Q^2}{3Mr-4Q^2}\right)-\Delta\frac{d^2}{dr^2}$$
$$-\left\{6(r-M)-\frac{4Q^2\Delta}{r(3Mr-4Q^2)}\right\}\frac{d}{dr}-4-\frac{2Q^2}{r^2}$$
$$+\frac{4Q^2(r^2+2Mr-3Q^2)}{r^2(3Mr-4Q^2)}+\frac{3Mr-4Q^2}{3Mr-2Q^2}(l-1)(l+2)\Big]R_l^{(2)} \quad \text{(C.1.17)}$$
$$=\frac{2\sqrt{2}Q\sqrt{(l-1)(l+2)}r^3}{3Mr-2Q^2}\Big(-i\omega\frac{r^2}{\Delta}+\frac{d}{dr}+\frac{4}{r}$$
$$-\frac{4Q^2}{r(3Mr-4Q^2)}\Big)R_l^{(1)}$$

$$\Big[-\omega^2\frac{r^4}{\Delta}+2i\omega r\left(-2+\frac{r(r-M)}{\Delta}-\frac{Q^2}{3Mr-2Q^2}\right)-\Delta\frac{d^2}{dr^2}$$
$$-\left\{\frac{6\Delta}{r}+4(r-M)-\frac{2Q^2\Delta}{r(3Mr-2Q^2)}\right\}\frac{d}{dr}-\frac{18r^2-24Mr+2Q^2}{r^2}$$
$$+\frac{12Q^2\Delta}{r^2(3Mr-2Q^2)}+\frac{3Mr-2Q^2}{3Mr-4Q^2}(l-1)(l+2)\Big]R_l^{(1)} \quad \text{(C.1.18)}$$
$$=\frac{-\sqrt{2}Q^2\sqrt{(l-1)(l+2)}\Delta}{r^3(3Mr-4Q^2)}\Big(i\omega\frac{r^2}{\Delta}+\frac{d}{dr}-\frac{2}{r}+\frac{4(r-M)}{\Delta}$$
$$-\frac{2Q^2}{r(3Mr-2Q^2)}\Big)R_l^{(2)}\ .$$

The quantities $(R_l^{(-1)})^*$ e $(R_l^{(-2)})^*$ (from χ^B_{-1} e Ψ^B_4), satisfy the same equations of $R_l^{(1)}$ and $R_l^{(2)}$. At this point decoupling this system of ordinary differential equations is straightforward.

Similarly, the Kerr case is separable using but the so-called spin-weighted spheroidal harmonics [8, 125]:

$$(H_0+H_1)\Theta(\theta)=-E\Theta(\theta) \quad \text{(C.1.19)}$$

where:

$$H_0=\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)-\left(\frac{m^2+s^2+2ms\cos\theta}{\sin^2\theta}\right)\right] \quad \text{(C.1.20)}$$

$$H_1=a^2\omega^2\cos^2\theta-2a\omega s\cos\theta \quad \text{(C.1.21)}$$

and E is the eigenvalue. We have factorized the spherical and the spheroidal parts to give the problem the form of a typical Quantum Mechanics exercise. In fact depending if the H_1 term is small or not, the way to approach the problem is very different. Unfortunately, in this case the laddering operators are not know [126] and this does not allow the same strategy used in the case of the Reissner-Nordström spacetime. In the case of the Kerr spacetime instead, this is not a problem because laddering operators are unnecessary to solve completely the problem. In the case of the Kerr-Newman spacetime this creates a "formal" problem. In fact the presence of the charge Q generates "ugly" terms which don't allow the separation of variables in all known coordinates. A hypothetical separation of variables in these coordinates would have been stopped by the explicit absence of laddering operators. During the last 25 years there have been various attempts to solve this problem. One idea, proposed in Chandrasekhar's monography [126], is to decouple the PDE's before the separation of variables, obtaining 4th order or higher linear PDE's. This task could be accomplished only using a super-computer, because of the 4th order derivatives. Another formulation was developed using de Cahen-Debever-Defrise formalism, but a part some elegant conservative equations, the problem has not been solved [127, 128]. In conclusion the problem remains still open. A new approach has been developed [9, 10] for vacuum spacetimes which gives directly the full set of perturbative equations. The direct extension of this work to the case of Einstein-Maxwell or more complicate spacetimes can put in a new light this difficult problem.

After this short historical overview we can discuss the results obtained by ICRANet researchers in this field. In [129], due to Cherubini and Ruffini, gravitational and electromagnetic perturbations to the Kerr-Newman spacetime using Maple tensor package are shown; a detailed analysis for slightly charged, rotating and oblate black hole is presented too. Subsequent to this article there have been various studies regarding the Teukolsky Master Equations (TMEs) in General Relativity. To this aim, a new form is found for the Teukolsky Master Equation in Kerr and interpreted in terms of de Rham-Lichenrowicz laplacians. The exact form of these generalized wave equations in any vacuum spacetime is given for the Riemann and Maxwell tensors, and the equations are linearized at any order, obtaining a hierarchy. It is shown that the TME for any Petrov type D spacetime is nothing more than a component of this laplacian linearized and that the TME cannot be derived by variational principles [9, 10]. More in detail, the Teukolsky Master Equation in the Kerr case, can be cast in a more compact form (Bini-Cherubini-Jantzen-

Ruffini form) by introducing a "connection vector" whose components are:

$$\begin{aligned}
\Gamma^t &= -\frac{1}{\Sigma}\left[\frac{M(r^2-a^2)}{\Delta} - (r + ia\cos\theta)\right] \\
\Gamma^r &= -\frac{1}{\Sigma}(r-M) \\
\Gamma^\theta &= 0 \\
\Gamma^\phi &= -\frac{1}{\Sigma}\left[\frac{a(r-M)}{\Delta} + i\frac{\cos\theta}{\sin^2\theta}\right] .
\end{aligned} \tag{C.1.22}$$

It's easy to prove that:

$$\nabla^\mu \Gamma_\mu = -\frac{1}{\Sigma} \quad , \quad \Gamma^\mu \Gamma_\mu = \frac{1}{\Sigma}\cot^2\theta + 4\psi_2^A \tag{C.1.23}$$

and consequently the Teukolsky Master Equation assumes the form:

$$[(\nabla^\mu + s\Gamma^\mu)(\nabla_\mu + s\Gamma_\mu) - 4s^2\psi_2^A]\psi^{(s)} = 4\pi T \tag{C.1.24}$$

where ψ_2^A is the only non vanishing NP component of the Weyl tensor in the Kerr background in the Kinnersley tetrad (C.1.5) (with $Q = 0$). Equation (C.1.24) gives a common structure for these massless fields in the Kerr background varying the "s" index. In fact, the first part in the lhs represents (formally) a D'Alembertian, corrected by taking into account the spin-weight, and the second part is a curvature (Weyl) term linked to the "s" index too. This particular form of the Teukolsky Master Equation forces us to extend this analysis in the next sections because it suggests a connection between the perturbation theory and a sort of generalized wave equations which differ from the standard ones by curvature terms. In fact generalized wave operators are know in the mathematical literature as De Rham-Lichnerowicz Laplacians and the curvature terms which make them different from the ordinary ones are given by the Weitzenböck formulas. Mostly known examples in electromagnetism are

- the wave equation for the vector potential A_μ:

$$\nabla_\alpha \nabla^\alpha A_\mu - R_\mu{}^\lambda A_\lambda = -4\pi J_\mu \quad , \quad \nabla^\alpha A_\alpha = 0 \tag{C.1.25}$$

- the wave equation for the Maxwell tensor :

$$\nabla^\mu \nabla_\mu F_{\nu\lambda} + R_{\rho\mu\nu\lambda} F^{\rho\mu} - R^\rho{}_\lambda F_{\nu\rho} + R^\rho{}_\nu F_{\lambda\rho} = -8\pi \nabla_{[\mu} J_{\nu]} \tag{C.1.26}$$

while for the gravitational case one has

- the wave equation for the metric perturbations:

$$\nabla_\alpha \nabla^\alpha \bar{h}_{\mu\nu} + 2R_{\alpha\mu\beta\nu}\bar{h}^{\alpha\beta} - 2R_{\alpha(\mu}\bar{h}_{\nu)}{}^{\alpha} = 0,$$

$$\nabla_\alpha \bar{h}_\mu{}^\alpha = 0, \qquad \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}g_{\mu\nu}h_\alpha{}^\alpha \tag{C.1.27}$$

- the wave equation for the Riemann Tensor

$$\begin{aligned} R^{\alpha\beta}{}_{\gamma\delta;\epsilon}{}^{\epsilon} &= 4R^{[\alpha}{}_{[\gamma;\delta]}{}^{\beta]} - 2R^{[\alpha}{}_{\epsilon}R^{\beta]\epsilon}{}_{\gamma\delta} - 2R^{\alpha\mu\beta\nu}R_{\mu\nu\gamma\delta} \\ &\quad -4R^{[\alpha}{}_{\mu\nu[\gamma}R^{\beta]\mu\nu}{}_{\delta]} \,. \end{aligned} \tag{C.1.28}$$

These equations are "non minimal," in the sense that they cannot be recovered by a minimal substitution from their flat space counterparts. A similar situation holds in the standard Quantum Field Theory for the electromagnetic Dirac equation. In fact, applying for instance to the Dirac equation an "ad hoc" first order differential operator one gets the second order Dirac equation

$$\begin{aligned} &(i\partial\!\!\!/ - eA\!\!\!/ + m)(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi = \\ &\left[(i\partial_\mu - eA_\mu)(i\partial^\mu - eA^\mu) - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} - m^2\right]\psi = 0, \end{aligned} \tag{C.1.29}$$

where the notation is obvious. It is easy to recognize in equation (C.1.29) a generalized Laplacian and a curvature (Maxwell) term applied to the spinor. Moreover this equation is "non minimal", in the sense that the curvature (Maxwell) term cannot be recovered by electromagnetic minimal substitution in the standard Klein-Gordon equation for the spinor components. The analogous second order Dirac equation in presence of a gravitational field also has a non minimal curvature term and reduces to the form:

$$(\nabla_\alpha \nabla^\alpha + m^2 + \frac{1}{4}R)\psi = 0 \quad . \tag{C.1.30}$$

The general TME formalism is applied to other exact solutions of the vacuum Einstein field equations of Petrov type D. A new analysis of the Kerr-Taub-NUT black hole is given, focussing on Mashhoon spin-coupling and superradiance [130, 59].

More in detail, in [130] Bini, Cherubini and Jantzen studied a single master equation describing spin $s = 0 - 2$ test field gauge and tetrad-invariant perturbations of the Taub-NUT spacetime. This solution of vacuum Einstein field equations describes a black hole with mass M and gravitomagnetic monopole moment ℓ. This equation can be separated into its radial and angular parts. The behaviour of the radial functions at infinity and near the horizon is studied. The angular equation, solved in terms of hypergeometric functions, can be related both to spherical harmonics of suitable weight, resulting from the coupling of the spin-weight of the field and the gravitomagnetic monopole moment of the spacetime, and to the total angular momentum operator associated with the spacetime's rotational symmetry. The results are compared with the Teukolsky master equation for the Kerr spacetime.

In [59] instead Bini, Cherubini, Jantzen and Mashhoon have studied a single master equation describing spin $s \leq 2$ test fields that are gauge- and tetrad-invariant perturbations of the Kerr-Taub-NUT (Newman - Unti - Tamburino) spacetime representing a source with a mass M, gravitomagnetic monopole moment $-\ell$, and gravitomagnetic dipole moment (angular momentum) per unit mass a. This equation can be separated into its radial and angular parts. The behavior of the radial functions at infinity and near the horizon is studied and used to examine the influence of l on the phenomenon of superradiance, while the angular equation leads to spin-weighted spheroidal harmonic solutions generalizing those of the Kerr spacetime. Finally, the coupling between the spin of the perturbing field and the gravitomagnetic monopole moment is discussed.

In [69] instead Bini and Cherubini investigate the algebraically special frequencies of Taub-NUT black holes in detail in comparison with known results concerning the Schwarzschild case. The periodicity of the time coordinate, required for regularity of the solution, prevents algebraically special frequencies to be physically acceptable. In the more involved Kerr-Taub-NUT case, the relevant equations governing the problem are obtained. The formalism is applied to the C-metric, and physical speculations are presented concerning the spin-acceleration coupling.

In [70] Bini, Cherubini and Mashhoon study the vacuum C metric and its physical interpretation in terms of the exterior spacetime of a uniformly accelerating spherically-symmetric gravitational source. Wave phenomena on the linearized C metric background are investigated. It is shown that the scalar perturbations of the linearized C metric correspond to the gravitational Stark effect. This effect is studied in connection with the Pioneer anomaly.

In [71] instead Bini, Cherubini and Mashhoon analysed the massless field perturbations of the accelerating Minkowski and Schwarzschild spacetimes. The results are extended to the propagation of the Proca field in Rindler spacetime. They examine critically the possibility of existence of a general spinacceleration coupling in complete analogy with the well-known spinrotation coupling. They argue that such a direct coupling between spin and linear acceleration does not exist.

In [72] Cherubini, Bini, Bruni and Perjes consider vacuum Kasner spacetimes, focusing on those that can be parametrized as linear perturbations of the special Petrov type D case. In particular they analyze in detail the perturbations which map the Petrov type D Kasner spacetime into another Kasner spacetime of Petrov type I. For these 'quasi-D' Kasner models they first investigate the modification to some curvature invariants and the principal null directions, both related to the Petrov classification of the spacetime. This simple Kasner example allows one to clarify the fact that perturbed spacetimes do not retain in general the speciality character of the background. In fact, there are four distinct principal null directions, although they are not necessarily first-order perturbations of the background principal null directions. Then in the Kasner type D background they derive a Teukolsky master equation, a classical tool for studying black-hole perturbations of any spin. This further step allows one to control totally general cosmologies around such a background as well as to show, from a completely new point of view, the well-known absence of gravitational waves in Kasner spacetimes.

C.2. Perturbations of a Reissner-Nordström black hole by a massive point charge at rest and the "electric Meissner effect"

The problem of the effect of gravity on the electromagnetic field of a charged particle leading to the consideration of the Einstein-Maxwell equations has been one of the most extensively treated in the literature, resulting in exact solutions (see [131] and references therein) as well as in a variety of approximation methods [132]-[140].

The issue of the interaction of a massive charged particle of mass m and charge q with a Reissner-Nordström black hole with mass $\mathcal{M}$ and charge Q has been addressed by the ICRANet collaboration: Bini, Geralico and Ruffini [141, 142, 143]. We have solved this problem by the first order perturbation approach formulated by Zerilli [120] using the tensor harmonic expansion of the Einstein-Maxwell system of equations.

The results discussed in [141, 142] gave answer to a problem whose investigation started long ago by Hanni and Ruffini [137]. They obtained the solution for a charged particle at rest in the field of a Schwarzschild black hole in the case of test field approximation, i.e. under the conditions $q/m \gg 1$, $m \approx 0$ and $q \ll \mathcal{M}$, $q \ll Q$, by using the vector harmonic expansion of the electromagnetic field in curved space. The conditions above imply the solution of the Maxwell equations only in a fixed Schwarzschild metric, since the perturbation to the background geometry given by the electromagnetic stress-energy tensor is second order in the particle's charge and the effect of the particle's mass is there neglected. As a result, no constraint on the position of the test particle follows from the Einstein equations and the Bianchi identities: the position of the particle is totally arbitrary.

This same test field approximation has been applied to the case of a Reissner-Nordström black hole by Leaute and Linet [140]. In analogy with the Schwarzschild case, they used the vector harmonic expansion of the electromagnetic field holding the background geometry fixed. However, this "test field approximation" is not valid in the present context. In fact, in addition to neglecting the effect of the particle mass on the background geometry, this treatment also neglects the electromagnetically induced gravitational perturbation terms linear in the charge of the particle which would contribute to modifying the metric as well.

The correct way to attack the problem is thus to solve the linearized Einstein-Maxwell equations following Zerilli's first order tensor harmonic analysis [120]. In fact the source terms of the Einstein equations comprise the energy-momentum tensor associated with the particle's mass, the electromagnetic energy-momentum tensor associated with the background field as well as additional interaction terms, to first order in m and q, proportional to the product of the square of the charge of the background geometry and the particle's mass ($\sim Q^2 m$) and to the product of the charges of both the particle and the black hole ($\sim qQ$). These terms give origin to the so called "electromagnetically induced gravitational perturbation" [144]. On the other hand, the source terms of the Maxwell equations contain the electromagnetic current associated with the particle's charge as well as interaction terms proportional to the product of the black hole's charge and the particle's mass ($\sim Qm$), giving origin to the "gravitationally induced electromagnetic perturbation" [145].

This has been explicitly done in [141, 142]. Let us briefly summarize the results and the properties of the solution derived there. In standard Schwarzschild-like coordinates the Reissner-Nordström black hole metric is

$$\begin{aligned} ds^2 &= -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \\ f(r) &= 1 - \frac{2\mathcal{M}}{r} + \frac{Q^2}{r^2}\,, \end{aligned} \tag{C.2.1}$$

with associated electromagnetic field

$$F = -\frac{Q}{r^2} dt \wedge dr\,. \tag{C.2.2}$$

The horizons are located at $r_\pm = \mathcal{M} \pm \sqrt{\mathcal{M}^2 - Q^2} \equiv \mathcal{M} \pm \Gamma$; we consider the case $|Q| \leq \mathcal{M}$ and the region $r > r_+$ outside the outer horizon, with an extremely charged hole corresponding to $|Q| = \mathcal{M}$ (which implies $\Gamma = 0$) where the two horizons coalesce.

The only nonvanishing components of the stress-energy tensor and of the current density are given by

$$\begin{aligned} T_{00}^{\rm part} &= \frac{m}{2\pi b^2} f(b)^{3/2} \delta(r-b)\,\delta(\cos\theta - 1) \\ J^0_{\rm part} &= \frac{q}{2\pi b^2} \delta(r-b)\,\delta(\cos\theta - 1)\ , \end{aligned} \tag{C.2.3}$$

which enter the system of combined Einstein-Maxwell equations

$$\begin{aligned} \tilde{G}_{\mu\nu} &= 8\pi \left(T^{\rm part}_{\mu\nu} + \tilde{T}^{\rm em}_{\mu\nu} \right)\ , \\ \tilde{F}^{\mu\nu}{}_{;\nu} &= 4\pi J^\mu_{\rm part}\ , \quad {}^*\tilde{F}^{\alpha\beta}{}_{;\beta} = 0\ . \end{aligned} \tag{C.2.4}$$

The quantities denoted by the tilde refer to the total electromagnetic and gravitational fields, to first order of the perturbation:

$$\begin{aligned} \tilde{g}_{\mu\nu} &= g_{\mu\nu} + h_{\mu\nu}\ , \qquad \tilde{F}_{\mu\nu} = F_{\mu\nu} + f_{\mu\nu}\ , \\ \tilde{T}^{\rm em}_{\mu\nu} &= \frac{1}{4\pi} \left[\tilde{g}^{\rho\sigma} \tilde{F}_{\rho\mu} \tilde{F}_{\sigma\nu} - \frac{1}{4} \tilde{g}_{\mu\nu} \tilde{F}_{\rho\sigma} \tilde{F}^{\rho\sigma} \right]\ , \\ \tilde{G}_{\mu\nu} &= \tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R}\ ; \end{aligned} \tag{C.2.5}$$

note that the covariant derivative operation makes use of the perturbed metric $\tilde{g}_{\mu\nu}$ as well. The corresponding quantities without the tilde refer to the background Reissner-Nordström geometry (C.2.1) and electromagnetic field (C.2.2). Following Zerilli's [120] procedure we expand the fields $h_{\mu\nu}$ and $f_{\mu\nu}$ as well as the source terms (C.2.3) in tensor harmonics, imposing then the Regge-Wheeler gauge [119] to simplify the description of the perturbation. The perturbation equations are then obtained from the system (C.2.4), keeping terms to first order in the mass m of the particle and its charge q which are assumed sufficiently small with respect to the black hole mass and charge. The axial symmetry of the problem about the z axis ($\theta = 0$) allows to put the azimuthal parameter equal to zero in the expansion, leading to a great simplification. Furthermore, it is sufficient to consider only electric-parity perturbations, since there are no magnetic sources [144, 145, 120].

The geometrical perturbations $h_{\mu\nu}$ for the electric multipoles in the Regge-Wheeler gauge are given by

$$||h_{\mu\nu}|| = \begin{bmatrix} e^\nu H_0 Y_{l0} & H_1 Y_{l0} & 0 & 0 \\ H_1 Y_{l0} & e^{-\nu} H_2 Y_{l0} & 0 & 0 \\ 0 & 0 & r^2 K Y_{l0} & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta K Y_{l0} \end{bmatrix}\ , \tag{C.2.6}$$

where Y_{l0} are normalized spherical harmonics with azimuthal index equal to zero and $e^{\nu} = f(r)$ is Zerilli's notation. The electromagnetic field harmonics $f_{\mu\nu}$ for the electric multipoles are given by

$$||f_{\mu\nu}|| = \begin{bmatrix} 0 & \tilde{f}_{01} Y_{l0} & \tilde{f}_{02} \frac{\partial Y_{l0}}{\partial \theta} & 0 \\ \text{antisym} & 0 & \tilde{f}_{12} \frac{\partial Y_{l0}}{\partial \theta} & 0 \\ \text{antisym} & \text{antisym} & 0 & 0 \\ \text{antisym} & \text{antisym} & \text{antisym} & 0 \end{bmatrix} , \tag{C.2.7}$$

where $\tilde{f}_{\mu\nu}$ denotes the θ-independent part of $f_{\mu\nu}$, and the symbol "antisym" indicates components obtainable by antisymmetry. The expansion of the source terms (C.2.3) gives the relations

$$\sum_l A_{00} Y_{l0} = 16\pi T_{00}^{\text{part}} , \qquad \sum_l v Y_{l0} = J^0_{\text{part}} , \tag{C.2.8}$$

with

$$A_{00} = 8\sqrt{\pi} \frac{m\sqrt{2l+1}}{b^2} f(b)^{3/2} \delta(r-b) , \; v = \frac{1}{2\sqrt{\pi}} \frac{q\sqrt{2l+1}}{b^2} \delta(r-b) . \tag{C.2.9}$$

The Einstein-Maxwell field equations (C.2.4) give rise to the following system of radial equations for values $l \geq 2$ of the multipoles (note that the cases $l = 0, 1$ must be treated separately)

$$\begin{aligned}
0 &= e^{2\nu} \left[2K'' - \frac{2}{r} W' + \left(\nu' + \frac{6}{r} \right) K' - 4 \left(\frac{1}{r^2} + \frac{\nu'}{r} \right) W \right] - \frac{2\lambda e^{\nu}}{r^2} (W + K) \\
&\quad -2 \frac{Q^2 e^{\nu} W}{r^4} - 4 \frac{Q e^{\nu} \tilde{f}_{01}}{r^2} + A_{00} , \\
0 &= \frac{2}{r} W' - \left(\nu' + \frac{2}{r} \right) K' - \frac{2\lambda e^{-\nu}}{r^2} (W - K) - 2 \frac{Q^2 e^{-\nu} W}{r^4} + 4 \frac{Q e^{-\nu} \tilde{f}_{01}}{r^2} , \\
0 &= K'' + \left(\nu' + \frac{2}{r} \right) K' - W'' - 2 \left(\nu' + \frac{1}{r} \right) W' \\
&\quad + \left(\nu'' + \nu'^2 + \frac{2\nu'}{r} \right) (K - W) - 2 \frac{Q^2 e^{-\nu} K}{r^4} + \frac{4 Q e^{-\nu}}{r^2} \tilde{f}_{01} , \\
0 &= -W' + K' - \nu' W + 4 \frac{Q e^{-\nu} \tilde{f}_{02}}{r^2} , \\
0 &= \tilde{f}_{01}{}' + \frac{2}{r} \tilde{f}_{01} - \frac{l(l+1) e^{-\nu} \tilde{f}_{02}}{r^2} - \frac{Q}{r^2} K' + 4\pi v , \\
0 &= \tilde{f}_{01} - \tilde{f}_{02}{}' ,
\end{aligned} \tag{C.2.10}$$

since $H_0 = H_2 \equiv W$, $H_1 \equiv 0$ and $\tilde{f}_{12} \equiv 0$, where $\lambda = \frac{1}{2}(l-1)(l+2)$ and a prime denotes differentiation with respect to r.

We have a system of 6 coupled ordinary differential equations for 4 unknown functions: K, W, $\tilde{f}_{01}$ and $\tilde{f}_{02}$. The compatibility of the system requires that these equations are not independent. Two equations can indeed be eliminated provided that the following stability condition holds

$$m = qQ\frac{bf(b)^{1/2}}{\mathcal{M}b - Q^2}\,, \tag{C.2.11}$$

involving the black hole and particle parameters as well as their separation distance b. If the black hole is extreme (i.e. $Q/\mathcal{M} = 1$), then the particle must also have the same ratio $q/m = 1$, and equilibrium exists independent of the separation. In the general non-extreme case $Q/\mathcal{M} < 1$ there is instead only one position of the particle which corresponds to equilibrium, for given values of the charge-to-mass ratios of the bodies. In this case the particle charge-to-mass ratio must satisfy the condition $q/m > 1$. Note that quite surprisingly Eq. (C.2.11) coincides with the equilibrium condition for a charged test particle in the field of a Reissner-Nordström black hole which has been discussed by Bonnor [146] in the simplified approach of test field approximation, neglecting all the feedback terms.

We then succeed in the exact reconstruction of both the perturbed gravitational and electromagnetic fields by summing all multipoles [142]. The perturbed metric is given by

$$\begin{aligned} d\tilde{s}^2 &= -[1 - \bar{\mathcal{H}} - k(r)]f(r)dt^2 + [1 + \bar{\mathcal{H}} + k(r)]f(r)^{-1}dr^2 \\ &\quad +(1 + \bar{\mathcal{H}})r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \\ k(r) &= \frac{\bar{\mathcal{H}}_0 Q^2}{r^2 f(r)}\,, \qquad \bar{\mathcal{H}}_0 = -2q\Gamma^2/[Q(\mathcal{M}b - Q^2)]\,, \end{aligned} \tag{C.2.12}$$

where

$$\begin{aligned} \bar{\mathcal{H}} &= 2\frac{m}{br}f(b)^{-1/2}\frac{(r-\mathcal{M})(b-\mathcal{M}) - \Gamma^2\cos\theta}{\bar{\mathcal{D}}}\,, \\ \bar{\mathcal{D}} &= [(r-\mathcal{M})^2 + (b-\mathcal{M})^2 - 2(r-\mathcal{M})(b-\mathcal{M})\cos\theta \\ &\quad -\Gamma^2\sin^2\theta]^{1/2}\,. \end{aligned} \tag{C.2.13}$$

It can be shown that this perturbed metric is spatially conformally flat; moreover, the solution remains valid as long as the condition $|\bar{\mathcal{H}}| \ll 1$ is satisfied. The total electromagnetic field to first order of the perturbation turns out to be

$$\tilde{F} = -\left[\frac{Q}{r^2} + E_r\right] dt \wedge dr - E_\theta dt \wedge d\theta\,, \tag{C.2.14}$$

with

$$
\begin{aligned}
E_r &= \frac{q}{r^3}\frac{\mathcal{M}r - Q^2}{\mathcal{M}b - Q^2}\frac{1}{\bar{\mathcal{D}}}\left\{\left[\mathcal{M}(b-\mathcal{M}) + \Gamma^2\cos\theta \right.\right.\\
&\quad \left. + \frac{Q^2[(r-\mathcal{M})(b-\mathcal{M}) - \Gamma^2\cos\theta]}{\mathcal{M}r - Q^2}\right]\\
&\quad \left. - \frac{r[(r-\mathcal{M})(b-\mathcal{M}) - \Gamma^2\cos\theta]}{\bar{\mathcal{D}}^2}[(r-\mathcal{M}) - (b-\mathcal{M})\cos\theta]\right\} ,\\
E_\theta &= q\frac{\mathcal{M}r - Q^2}{\mathcal{M}b - Q^2}\frac{b^2 f(b) f(r)}{\bar{\mathcal{D}}^3}\sin\theta \,. \qquad \text{(C.2.15)}
\end{aligned}
$$

Note that in the extreme case $Q/\mathcal{M} = q/m = 1$ this solution reduces to the linearized form of the well known exact solution by Majumdar and Papapetrou [147, 148] for two extreme Reissner-Nordström black holes. Furthermore, this solution satisfies Gauss' theorem

$$
\Phi = \int_S {}^*\tilde{F} \wedge dS = 4\pi[Q + q\vartheta(r-b)] \,, \qquad \text{(C.2.16)}
$$

where Φ is the flux of the electric field obtained by integrating the dual of the electromagnetic form (C.2.14) over a spherical 2-surface S centered at the origin where the black hole charge Q is placed and with variable radius (r greater or lesser than b), the function $\vartheta(x)$ denoting the step function.

Recently an important progress has been achieved by Belinski and Alekseev [149]. They have obtained an exact two-body solution of the Einstein-Maxwell equations in explicit analytic form for the system consisting of a Reissner-Nordström black hole and a naked singularity, by using the monodromy transform approach [150]. They have shown that an equilibrium without intervening struts or tensions is possible for such a system at selected values of the separating distance between the sources. Furthermore, their equilibrium condition exactly reduces to our equation (C.2.11) once linearized with respect to the mass and charge of the naked singularity. We have indeed been able to show explicitly the coincidence between the linearized form of their exact solution and our perturbative solution.

We have then analyzed in [143] the properties of the perturbed electric field with special attention to the construction of the lines of force of the electric field. The two cases have been considered of the sole particle, with the subtraction of the dominant contribution of the black hole, as well as of the total field due to the black hole and the particle. As the black hole becomes extreme an effect similar to the ordinary Meissner effect for magnetic fields in the presence of superconductors arises: the electric field lines of the point charge are expelled outside the outer horizon. Note that this effective "electric Meissner effect" has no classical analogue, as far as we know, and is a pure general relativistic effect. Let us discuss this issue more in detail.

The electric field lines are defined as the solution of the differential equation

$$\frac{dx^\alpha}{d\lambda} = E(U)^\alpha \,, \tag{C.2.17}$$

where λ is an affine parameter for the lines and $E(U)^\alpha$ are the coordinate components of the electric field

$$E(U)^\alpha = F^\alpha{}_\beta U^\beta \tag{C.2.18}$$

as measured by an observer with four-velocity U. The shape of the lines thus depends on the choice of the observer and that of the coordinates which are used to draw the curves. We refer to the static observers with respect to the metric (C.2.12), whose four-velocity is given by

$$U = \frac{1}{\sqrt{-\tilde{g}_{tt}}}\partial_t = f(r)^{-1/2}\left(1 + \frac{\bar{\mathcal{H}} + k(r)}{2}\right)\partial_t \,, \tag{C.2.19}$$

to first order of the perturbation. Eq. (C.2.17) thus becomes

$$\frac{dr}{d\lambda} = E(U)^r \,, \qquad \frac{d\theta}{d\lambda} = E(U)^\theta \,, \tag{C.2.20}$$

leading to the equation

$$-\,E(U)^r\, d\theta + E(U)^\theta\, dr = 0 \,, \tag{C.2.21}$$

after eliminating the parameter λ.

For a static spacetime and using a static family of observers the electric lines of force coincide with the constant flux lines [151]. The flux across a generic 2-surface S is given by

$$\Phi = \int_S \left[{}^*\tilde{F}_{r\phi} dr\, d\phi + {}^*\tilde{F}_{\theta\phi} d\theta\, d\phi\right] \,, \tag{C.2.22}$$

since the only nonvanishing components of ${}^*\tilde{F}$ are

$$\begin{aligned} {}^*\tilde{F}_{\theta\phi} &= -r^2 \sin\theta \left[-(1+\bar{\mathcal{H}})\frac{Q}{r^2} + f_{tr}\right] \equiv {}^*\tilde{F}^{(0)}_{\theta\phi} + {}^*\tilde{F}^{(1)}_{\theta\phi} \,, \\ {}^*\tilde{F}_{r\phi} &= f(r)^{-1} \sin\theta f_{t\theta} \equiv {}^*\tilde{F}^{(1)}_{r\phi} \,, \end{aligned} \tag{C.2.23}$$

where the superscripts (0), (1) refer to the zeroth order and first order terms respectively. Therefore, as the electromagnetic field components do not depend explicitly on ϕ, if S is a generic revolution surface around the symmetry

z-axis we can write

$$\Phi = 2\pi \int_S \left[{}^*\tilde{F}_{r\phi} dr + {}^*\tilde{F}_{\theta\phi} d\theta \right] , \tag{C.2.24}$$

so that the elementary flux across an infinitesimal surface (closed, limited by the two spherical caps: $\phi \in [0, 2\pi]$, $\theta = \theta_0$ and $r = r_0$ and $\phi \in [0, 2\pi]$, $\theta = \theta_0 + d\theta$ and $r = r_0 + dr$) of this kind is

$$d\Phi = 2\pi [{}^*\tilde{F}_{r\phi} dr + {}^*\tilde{F}_{\theta\phi} d\theta] . \tag{C.2.25}$$

The lines of constant electric flux ($d\Phi = 0$) are then defined as those curves solutions of the equation

$$0 = {}^*\tilde{F}_{r\phi} dr + {}^*\tilde{F}_{\theta\phi} d\theta , \tag{C.2.26}$$

which coincides with Eq. (C.2.21), noting that

$${}^*\tilde{F}_{\theta\phi} = -\frac{\sqrt{-\tilde{g}}}{U_0} E(U)^r , \qquad {}^*\tilde{F}_{r\phi} = \frac{\sqrt{-\tilde{g}}}{U_0} E(U)^\theta . \tag{C.2.27}$$

We are now ready to draw the electric lines of force by numerically integrating Eq. (C.2.21). Note that if the total electric field is considered the contribution of the black hole always dominates (see Fig. C.1).

We are mainly interested in studying the "effective field" representing the net effect of the perturbation induced by the massive charged particle on the background electric field. The most natural way to separate the two contributions is to directly use the elementary flux equation (C.2.25). By requiring that the integration over a spherical 2-surface S centered at the origin gives the first order contribution $\Phi^{(1)} = 4\pi q \vartheta(r - b)$ only to the total electric flux (C.2.16), i.e. the charge of the particle only, we get

$$d\Phi^{(1)} = 2\pi [{}^*\tilde{F}^{(1)}_{r\phi} dr + {}^*\tilde{F}^{(1)}_{\theta\phi} d\theta] . \tag{C.2.28}$$

The "effective field" lines corresponding to the perturbation with the contribution of the black hole electric field being subtracted are thus defined as the lines of constant flux $d\Phi^{(1)} = 0$, namely

$$0 = {}^*\tilde{F}^{(1)}_{r\phi} dr + {}^*\tilde{F}^{(1)}_{\theta\phi} d\theta , \tag{C.2.29}$$

according to Eq. (C.2.26), which reduces to the previous one when only the contribution of those terms which are first order is taken into account.

The behavior of the lines of force of the effective electric field of the particle alone is shown in Fig. C.2.

Following Hanni and Ruffini [137] we now compute the induced charge on

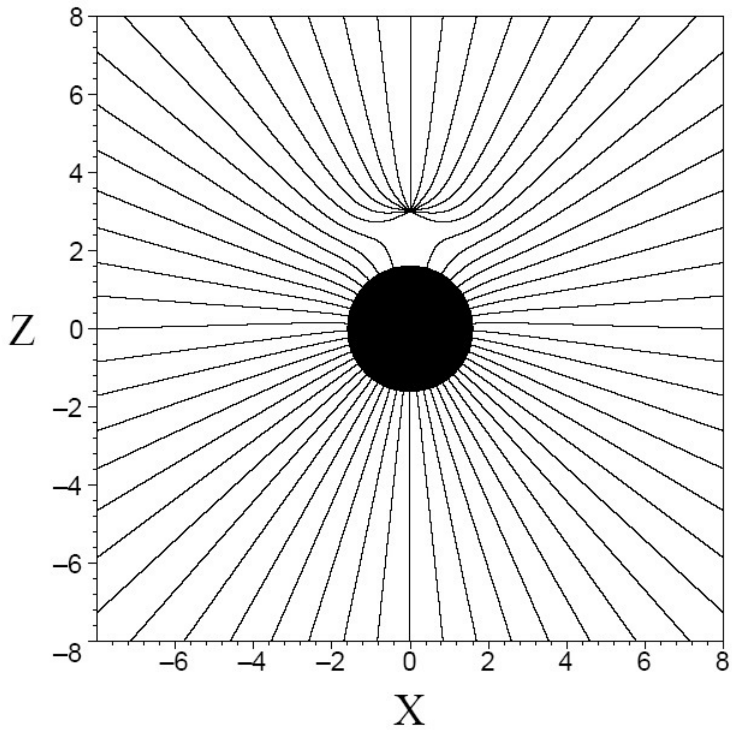

Figure C.1.: Lines of force of the total electric field of the black hole and particle in the $X - Z$ plane ($X = r\sin\theta$, $Z = r\cos\theta$ are Cartesian-like coordinates) for charges of the same sign $q/Q = 0.1$ and fixed parameter values $b/\mathcal{M} = 3$ and $Q/\mathcal{M} = 0.8$.

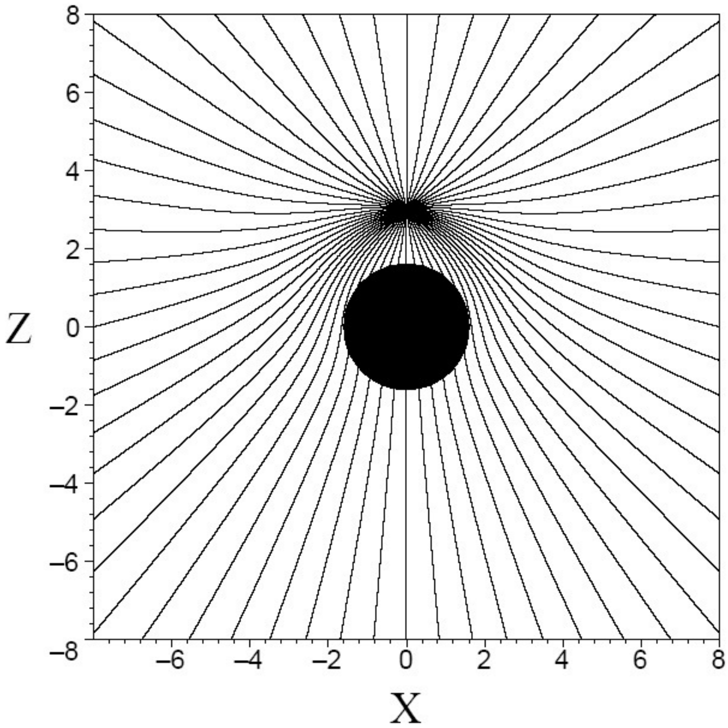

Figure C.2.: Lines of force of the effective electric field of the sole particle in the non-extreme case for the same choice of parameters as in Fig. C.1. As explained in the text this "effective field" is obtained by subtracting the dominant contribution of the black hole own electric field to the total perturbed field, thus representing the net effect of the perturbation induced by the massive charged particle on the background field.

the surface of the black hole horizon. Some lines of force intersect the horizon. If the particle is positively charged, at angles smaller than a certain critical angle the induced charge is negative and the lines of force cross the horizon. At angles greater than the critical angle the induced charge is positive and the lines of force extend out of the horizon. At the critical angle the induced charge density vanishes and the lines of force of the electric field are tangent to the horizon. The total electric flux through the horizon and thus the total induced charge are zero.

The induced charge density on the horizon $\sigma^H(\theta)$ is defined in such a way that the amount of induced charge on an infinitesimal portion of the horizon sphere $r = r_+$ between $\theta = \theta_0$ and $\theta = \theta_0 + d\theta$ equals $1/(4\pi)$ times the elementary flux across the same surface

$$\frac{1}{4\pi} d\Phi|_{r_+} = \frac{1}{4\pi} 2\pi {}^*\tilde{F}^{(1)}_{\theta\phi}\big|_{r_+} d\theta = 2\pi r_+^2 \sigma^H(\theta) \sin\theta d\theta \,, \tag{C.2.30}$$

implying

$$\frac{{}^*\tilde{F}^{(1)}_{\theta\phi}\big|_{r_+}}{r_+^2 \sin\theta} = 4\pi\sigma^H(\theta) \,. \tag{C.2.31}$$

This can be identified with the surface version of the Gauss' law. The corresponding expression for the critical angle $\theta_{(\mathrm{crit})}$ comes from the condition $\sigma^H(\theta_{(\mathrm{crit})}) = 0$. Hence it results

$$\begin{aligned} \sigma^H(\theta) &= \frac{q}{4\pi r_+} \frac{\Gamma^2}{\mathcal{M}b - Q^2} \\ &\quad \times \frac{\Gamma(1+\cos^2\theta) - 2(b-\mathcal{M})\cos\theta}{[b - \mathcal{M} - \Gamma\cos\theta]^2} \,, \end{aligned} \tag{C.2.32}$$

$$\theta_{(\mathrm{crit})} = \arccos\left[\frac{b - \mathcal{M} - \sqrt{(b-\mathcal{M})^2 - \Gamma^2}}{\Gamma}\right] \,. \tag{C.2.33}$$

Assuming then the black hole and particle both have positive charge, one can evaluate the total amount of negative charge induced on the horizon by the particle

$$\begin{aligned} Q^{(-)}_{\mathrm{ind}} &= \int_\Sigma \sigma^H(\theta) d\Sigma = 2\pi r_+^2 \int_0^{\theta_{(\mathrm{crit})}} \sigma^H(\theta) \sin\theta d\theta \\ &= -q \frac{\Gamma r_+}{\mathcal{M}b - Q^2} \cos\theta_{(\mathrm{crit})} \,, \end{aligned} \tag{C.2.34}$$

where $d\Sigma = \sqrt{g_{\theta\theta} g_{\phi\phi}}\, d\theta\, d\phi$ and Σ is the spherical cap $0 \le \theta \le \theta_{(\mathrm{crit})}$.

Let us study what happens as the black hole approaches the extreme condition $|Q| = \mathcal{M}$ (implying $\Gamma = 0$). Eq. (C.2.32) shows that the induced charge

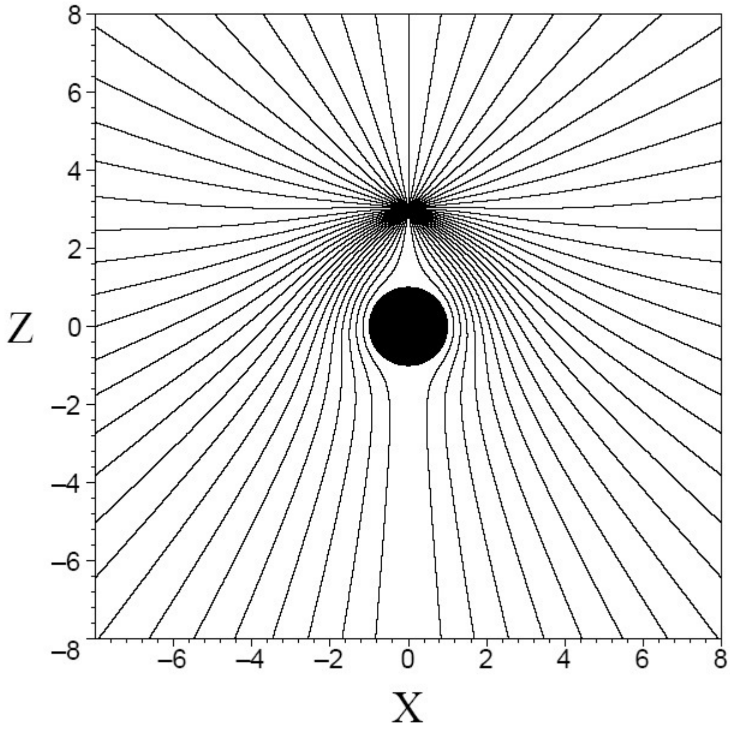

Figure C.3.: Lines of force of the effective electric field of the particle alone in the extreme case $Q/\mathcal{M} = 1$ for the same choice of the distance parameter as in Fig. C.1. No lines of force intersect the black hole horizon in this case, leading to the the "electric Meissner effect."

density on the horizon degenerates to zero for every value of the angle θ; the critical angle (C.2.33) approaches the value $\pi/2$ and the amount of negative charge (C.2.34) induced on the horizon vanishes identically. Therefore no lines of force cross the horizon, remaining tangent to it for every value of the polar angle, since every angle becomes critical: as the black hole approaches the extreme condition the electric field lines are thus pulled off the outer horizon and never intersect it when the black hole becomes extreme. The situation is summarized in Fig. C.3 showing the behavior of the lines of force of the effective electric field of the particle alone in the extreme case.

The "electric Meissner effect" above described is suitable to a suggestive interpretation in terms of the nature of the Reissner-Nordström solution. As soon as the black hole is not extreme the point particle induces charge on the horizon, and accordingly the electric field lines terminate on it; when the black hole becomes extreme no further charge induction is possible (unless one turns the black hole into a naked singularity), and coherently the electric field lines no more cross the horizon. In a sense the black hole rejects to turn itself into a naked singularity and this might be thought of as an argument in favor of the Cosmic Censorship conjecture.

C.3. Perturbations of a de Sitter spacetime

Let us consider the perturbations

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \tag{C.3.1}$$

of the de Sitter spacetime,

$$g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + N^{-2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\theta^2) \tag{C.3.2}$$

where $N^2 = (1 - H^2r^2)$, H being the Hubble constant. Since the background metric is static and spherically symmetric, we decompose the metric perturbation $h_{\mu\nu}$ in tensor harmonics and Fourier transform it with respect to time, as customary. We will use the Regge-Wheeler [119] gauge to simplify the form of the perturbation.

C.3.1. Electric multipoles

The metric perturbation $h_{\mu\nu}$, for the electric multipoles, is given by

$$||h_{\mu\nu}|| = \begin{bmatrix} N^2H_0 & H_1 & 0 & 0 \\ \text{sym} & \frac{1}{N^2}H_2 & 0 & 0 \\ \text{sym} & \text{sym} & r^2K & 0 \\ \text{sym} & \text{sym} & \text{sym} & r^2\sin^2\theta K \end{bmatrix} e^{-i\omega t}Y_{l0}\,, \tag{C.3.3}$$

where the symbol "sym" indicates that the missing components of $h_{\mu\nu}$ should be found from the symmetry $h_{\mu\nu} = h_{\nu\mu}$ and the functions Y_{l0} are normalized spherical harmonics with azimuthal index $m = 0$, defined by

$$Y_{l0} = \frac{1}{2}\sqrt{\frac{2l+1}{\pi}}P_l(\cos\theta)\,. \tag{C.3.4}$$

For $l \geq 2$, the system of radial equations we have to solve is the following:

$$\begin{aligned} 0 &= {H_1}' + \frac{i\omega}{N^2}(W+K) - \frac{2rH^2}{N^2}H_1\,, \\ 0 &= K' - \frac{W}{r} - \frac{iL}{2\omega r^2}H_1 + \frac{K}{rN^2}\,, \end{aligned} \tag{C.3.5}$$

where $L = l(l+1)$, since

$$H_0 = H_2 \equiv W\,, \tag{C.3.6}$$

together with the algebraic relation

$$0 = \frac{L-2}{r}W - \frac{i}{\omega}(H^2L + 2\omega^2)H_1 + \frac{2}{rN^2}\left(1 + r^2\omega^2 - N^2\frac{L}{2}\right)K\,. \tag{C.3.7}$$

Let us introduce the dimensionless variables $x = Hr$, $\Omega = \omega/H$ and denote $H_1 = i\tilde{H}_1$. Solving for W in the constraint equation (C.3.7) and substituting into the system (C.3.5) we get

$$\begin{aligned}
\tilde{H}_1{}' &= \frac{(2\Omega^2+3L-4)x}{(1-x^2)(L-2)}\tilde{H}_1 + \frac{2\Omega[(\Omega^2+L-1)x^2+2-L]}{(1-x^2)^2(L-2)}K\,,\\
K' &= -\frac{[2x^2(L+2\Omega^2)+L(L-2)]}{2(L-2)\Omega x^2}\tilde{H}_1 - \frac{x(L+2\Omega^2)}{(1-x^2)(L-2)}K\,,
\end{aligned} \tag{C.3.8}$$

where now a prime denotes differentiation with respect to x, which can be solved in terms of Heun's functions.

A real solution for the metric can be obtained by considering W and K and $\tilde{H}_1$ as real. In this case the nonvanishing metric components are

$$\begin{aligned}
g_{tt} &= -N^2(1 - WY_{l0}\cos\omega t)\\
g_{tr} &= \tilde{H}_1 Y_{l0}\sin\omega t\\
g_{rr} &= N^{-2}(1 + WY_{l0}\cos\omega t)\\
g_{\theta\theta} &= r^2(1 + KY_{l0}\cos\omega t)\\
g_{\phi\phi} &= r^2\sin^2\theta(1 + KY_{l0}\cos\omega t)\,,
\end{aligned} \tag{C.3.9}$$

so that

$$\begin{aligned}
ds^2 &= -N^2(1 - WY_{l0}\cos\omega t)\left(dt + \tilde{H}_1 N^{-2}Y_{l0}\sin\omega t\,dr\right)^2\\
&\quad + N^{-2}(1 + WY_{l0}\cos\omega t)dr^2\\
&\quad + r^2(1 + KY_{l0}\cos\omega t)(d\theta^2 + \sin^2\theta d\phi^2)\,,
\end{aligned} \tag{C.3.10}$$

at first order in the perturbation quantities. A natural orthonormal frame associated with this form of the metric is then

$$\begin{aligned}
\Omega^{\hat{0}} &= N\left(1-\frac{1}{2}WY_{l0}\cos\omega t\right)\left(dt-\tilde{H}_1N^{-2}Y_{l0}\sin\omega t\,dr\right)\\
&= N\left(1-\frac{1}{2}WY_{l0}\cos\omega t\right)dt-\tilde{H}_1N^{-1}Y_{l0}\sin\omega t\,dr\\
\Omega^{\hat{r}} &= N^{-1}\left(1+\frac{1}{2}WY_{l0}\cos\omega t\right)dr\\
\Omega^{\hat{\theta}} &= r\left(1+\frac{1}{2}KY_{l0}\cos\omega t\right)d\theta\\
\Omega^{\hat{\phi}} &= r\sin\theta\left(1+\frac{1}{2}KY_{l0}\cos\omega t\right)d\phi\,. \qquad \text{(C.3.11)}
\end{aligned}$$

One can then introduce a NP in a standard way

$$l=(\Omega^{\hat{0}}+\Omega^{\hat{r}})/\sqrt{2}\,,\quad n=(\Omega^{\hat{0}}-\Omega^{\hat{r}})/\sqrt{2}\,,\quad m=(\Omega^{\hat{\theta}}+i\Omega^{\hat{\phi}})/\sqrt{2}\,. \qquad \text{(C.3.12)}$$

C.3.2. Magnetic multipoles

The metric perturbation $h_{\mu\nu}$, for the magnetic multipoles, is given by

$$||h_{\mu\nu}||=\begin{bmatrix} 0 & 0 & 0 & h_0\\ \text{sym} & 0 & 0 & h_1\\ \text{sym} & \text{sym} & 0 & 0\\ \text{sym} & \text{sym} & \text{sym} & 0\end{bmatrix} e^{-i\omega t}\sin\theta\frac{dY_{l0}}{d\theta}\,. \qquad \text{(C.3.13)}$$

For $l\geq 2$, the system of radial equations we have to solve is the following:

$$\begin{aligned}
0 &= h_0'-\frac{2h_0}{r}+ih_1\left[\omega-\frac{N^2}{r^2}\frac{L-2}{\omega}\right]\,,\\
0 &= h_1'-\frac{2rH^2}{N^2}h_1-\frac{i\omega}{N^4}h_0\,, \qquad \text{(C.3.14)}
\end{aligned}$$

where $L=l(l+1)$.

By introducing the dimensionless variables $x=Hr$, $\Omega=\omega/H$ as above and setting $h_0=i\tilde{h}_0$, the system (C.3.14) becomes

$$\begin{aligned}
\tilde{h}_0' &= \frac{2}{x}\tilde{h}_0-\frac{(\Omega^2+L-2)x^2+2-L}{\Omega x^2}h_1\,,\\
h_1' &= \frac{\Omega}{(1-x^2)^2}\tilde{h}_0+\frac{2x}{1-x^2}h_1\,, \qquad \text{(C.3.15)}
\end{aligned}$$

where now a prime denotes differentiation with respect to x, which can be solved in terms of Heun's functions.

A real solution for the metric can be obtained by considering $\tilde{h}_0$ and h_1 as real. In this case the nonvanishing metric components are

$$\begin{aligned} g_{tt} &= -N^2\,, \quad g_{t\phi} = \tilde{h}_0 \sin\omega t \sin\theta \frac{dY_{l0}}{d\theta}\,, \quad g_{r\phi} = h_1 \cos\omega t \sin\theta \frac{dY_{l0}}{d\theta}\,, \\ g_{rr} &= N^{-2}\,, \quad g_{\theta\theta} = r^2\,, \quad g_{\phi\phi} = r^2 \sin^2\theta\,, \end{aligned} \tag{C.3.16}$$

so that

$$ds^2 = ds^2_{(\mathrm{dS})} + 2\sin\theta \frac{dY_{l0}}{d\theta} \left(\tilde{h}_0 \sin\omega t d\phi + 2h_1 \cos\omega t dr\right) dt\,, \tag{C.3.17}$$

at first order in the perturbation quantities. A natural orthonormal frame associated with this form of the metric is then

$$\begin{aligned} \Omega^{\hat{0}} &= N dt + \frac{\sin\theta}{N} \frac{dY_{l0}}{d\theta} (\tilde{h}_0 \sin\omega t d\phi + h_1 \cos\omega t dr) \\ \Omega^{\hat{r}} &= N^{-1} dr\,, \quad \Omega^{\hat{\theta}} = r d\theta\,, \quad \Omega^{\hat{\phi}} = r \sin\theta d\phi\,. \end{aligned} \tag{C.3.18}$$

One can then introduce a NP in a standard way, as in Eq. (C.3.12).

Writing closed form expressions for the electric and magnetic perturbations allows for the full reconstruction of the metric, a fact of fundamental importance to proceed with the comparison with curvature perturbations as well as to perform a systematic analysis of gauge conditions.

D. Cosmology

The Bianchi type IX spatially homogeneous vacuum spacetime also known as the Mixmaster universe has served as a theoretical playground for many ideas in general relativity, one of which is the question of the nature of the chaotic behavior exhibited in some solutions of the vacuum Einstein equations and another is the question of whether or not one can interpret the spacetime as a closed gravitational wave. In particular, to describe the mathematical approach to an initial cosmological singularity, the exact Bianchi type IX dynamics leads to the BLK approximation involving the discrete BLK map which acts as the transition between phases of approximately Bianchi type I evolution. The parameters of this map are not so easily extracted from the numerical evolution of the metric variables. However, recently it has been realized that these parameters are directly related to transitions in the scale-free part of the Weyl tensor. In fact this leads to a whole new interpretation of what the BLK dynamics represents.

For a given foliation of any spacetime, one can always introduce the scale free part of the extrinsic curvature when its trace is nonzero by dividing by that trace. In the expansion-normalized approach to spatially homogeneous dynamics, this corresponds to the expansion-normalized gravitational velocity variables. This scale free extrinsic curvature tensor can be characterized by its eigenvalues, whose sum is 1 by definition: these define three functions of the time parametrizing the foliation which generalize the Kasner indices of Bianchi type I vacuum spacetimes. A phase of velocity-dominated evolution is loosely defined as an interval of time during which the spatial curvature terms in the spacetime curvature are negligible compared to the extrinsic curvature terms. Under these conditions the vacuum Einstein equations can be approximated by ordinary differential equations in the time. These lead to a simple scaling of the eigenvectors of the extrinsic curvature during which the generalized Kasner indices remain approximately constant and simulate the Bianchi type I Kasner evolution.

The Weyl tensor can be also be repackaged as a second rank but complex spatial tensor with respect to the foliation and its scale free part is determined by a single complex scalar function of its eigenvalues, a number of particular representations for which are useful. In particular the so called speciality index is the natural choice for this variable which is independent of the permutations of the spatial axes used to order the eigenvalues, and so is a natural 4-dimensional tracker of the evolving gravitational field quotienting out all 3-dimensional gauge-dependent quantities. During a phase of velocity-dominated ("Kasner") evolution, the Weyl tensor is approximately determined by the extrinsic curvature alone, and hence the scalefree invariant part of the Weyl tensor is locked to the generalized Kasner indices exactly as in a Kasner spacetime. Of course during transitions between velocity-dominated evolution where the spatial curvature terms are important, the generalized Kasner indices and the Weyl tensor are uncoupled in their evolution, but the transition between one set of generalized Kasner indices and the next is locked to a transition in the scalefree Weyl tensor. This idealized mapping, approximated by the BKL map between Kasner triplets, can be reinterpreted as a continuous transition in the Weyl tensor whose scale invariant part can be followed through the transition directly. For spatially homogeneous vacuum spacetimes, the BLK transition is a consequence of a Bianchi type II phase of the dynamics which can be interpreted as a single bounce with a curvature wall in the Hamiltonian approach to the problem. One can in fact follow this transition in the Weyl tensor directly with an additional first order differential equation which is easily extracted from the Newman-Penrose equations expressed in a frame adapted both to the foliation and the Petrov type of the Weyl tensor.

This type of Weyl transition in the Mixmaster dynamics can be followed approximately using the Bianchi type II approximation to a curvature bounce, leading to a temporal spike in the real and imaginary parts of the speciality index which represents a circuit in the complex plane between the two real asymptotic Kasner points (a "pulse"). The graph of the speciality index versus time thus serves as a sort of electrocardiogram of the "heart" of the Mixmaster dynamics, stripping away all the gauge and frame dependent details of its evolution except for the choice of time parametrization, which is a recent nice result of our investigation.

E. Exact solutions

In this appendix we briefly review a recent work on the Kerr-Schild ansatz used to search for new exact solutions of Einstein's field equations. Moreover, we introduce a more recent work on exact solution with stationarity and axial symmetry and with metric functions as rational functions of the non ignorable coordinates. A more detailed discussion of both these works can be found in the section "The Kerr-Newman solution" of the present report.

E.1. Kerr-Schild metrics and the Kerr-Schild ansatz revisited

Kerr-Schild metrics have the form [152, 153]

$$\mathrm{d}s^2 = g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta \equiv (\eta_{\alpha\beta} - 2Hk_\alpha k_\beta)\mathrm{d}x^\alpha \mathrm{d}x^\beta \,, \tag{E.1.1}$$

where $\eta_{\alpha\beta}$ is the metric for Minkowski space and k_α is a null vector

$$\eta_{\alpha\beta}k^\alpha k^\beta = g_{\alpha\beta}k^\alpha k^\beta = 0, \qquad k^\alpha = \eta^{\alpha\beta}k_\beta = g^{\alpha\beta}k_\beta \,. \tag{E.1.2}$$

The inverse metric is also linear in H

$$g^{\alpha\beta} = \eta^{\alpha\beta} + 2Hk^\alpha k^\beta \,, \tag{E.1.3}$$

and so the determinant of the metric is independent of H

$$(\eta_{\alpha\gamma} - 2Hk_\alpha k_\gamma)(\eta^{\gamma\beta} + 2Hk^\gamma k^\beta) = \delta^\beta_\alpha \quad \longrightarrow \quad |g_{\alpha\beta}| = |\eta_{\alpha\beta}| \,.$$

Within this class of general metrics the Kerr solution was obtained in 1963 by a systematic study of algebraically special vacuum solutions [154]. If $(x^0 = t, x^1 = x, x^2 = y, x^3 = z)$ are the standard Cartesian coordinates for Minkowski spacetime with $\eta_{\alpha\beta} = \mathrm{diag}[-1, 1, 1, 1]$, then for Kerr metric we have

$$-k_\alpha \mathrm{d}x^\alpha = \mathrm{d}t + \frac{(rx + ay)\mathrm{d}x + (ry - ax)\mathrm{d}y}{r^2 + a^2} + \frac{z}{r}\mathrm{d}z \,, \tag{E.1.4}$$

where r and H are defined implicitly by

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1 \,, \qquad H = -\frac{\mathcal{M}r^3}{r^4 + a^2 z^2} \,. \tag{E.1.5}$$

Kerr solution is asymptotically flat and the constants $\mathcal{M}$ and a are the total mass and specific angular momentum for a localized source. They both have the dimension of a length in geometrized units. The vector $\mathbf{k}$ is geodesic and shearfree, implying that Kerr metric is algebraically special according to the Goldberg-Sachs theorem [155]. Moreover, $\mathbf{k}$ is independent of $\mathcal{M}$ and hence a function of a alone. Note that the mass parameter $\mathcal{M}$ appears linearly in the metric, i.e. in H.

We consider here Kerr-Schild metrics (E.1.1) as *exact linear perturbations* of Minkowski space and solve Einstein's field equations order by order in powers of H. The results of this analysis will be that $\mathbf{k}$ must be geodesic and shearfree as a consequence of third and second order equations, leading to an alternative derivation of Kerr solution.

E.1.1. Modified ansatz

Let ϵ be an arbitrary constant parameter, eventually to be set equal to 1, so that the Kerr-Schild metric (E.1.1) reads

$$g_{\alpha\beta} = \eta_{\alpha\beta} - 2\epsilon H k_\alpha k_\beta \,, \tag{E.1.6}$$

with inverse

$$g^{\alpha\beta} = \eta^{\alpha\beta} + 2\epsilon H k^\alpha k^\beta \,, \tag{E.1.7}$$

and suppose that coordinates are chosen so that the components $\eta_{\alpha\beta}$ are constants, but not necessarily of the form $\eta_{\alpha\beta} = \mathrm{diag}[-1,1,1,1]$. The connection is then quadratic in ϵ

$$\Gamma^\gamma{}_{\alpha\beta} = \epsilon \underset{1}{\Gamma}{}^\gamma{}_{\alpha\beta} + \epsilon^2 \underset{2}{\Gamma}{}^\gamma{}_{\alpha\beta} \,,$$

where

$$\begin{aligned} \underset{1}{\Gamma}{}^\gamma{}_{\alpha\beta} &= -(Hk_\alpha k^\gamma)_{,\beta} - (Hk_\beta k^\gamma)_{,\alpha} + (Hk_\alpha k_\beta)_{,\lambda}\eta^{\lambda\gamma} \,, && \text{(E.1.8)} \\ \underset{2}{\Gamma}{}^\gamma{}_{\alpha\beta} &= 2H[H(\dot{k}_\alpha k_\beta + \dot{k}_\beta k_\alpha) + \dot{H} k_\alpha k_\beta]k^\gamma \equiv 2Hk^\gamma (Hk_\alpha k^\beta)^{\cdot} \,, && \text{(E.1.9)} \end{aligned}$$

where a "dot" denotes differntiation in the $\mathbf{k}$ direction, i.e. $\dot{f} = \mathbf{k}(f) = f_{,\alpha}k^\alpha$. Note that only the indices of $\mathbf{k}$ can be raised and lowered with the Minkowski metric. Hereafter we will use an "index" 0 to denote contraction with $\mathbf{k}$, i.e.

$$\begin{aligned} \Gamma^0{}_{\alpha\beta} &= \Gamma^\gamma{}_{\alpha\beta}k_\gamma = \epsilon(Hk_\alpha k_\beta)^{\cdot} \,, && \text{(E.1.10)} \\ \Gamma^\gamma{}_{\alpha 0} &= \Gamma^\gamma{}_{\alpha\beta}k^\beta = -\epsilon(Hk_\alpha k^\gamma)^{\cdot} \,, && \text{(E.1.11)} \\ \Gamma^\gamma{}_{00} &= \Gamma^\gamma{}_{\alpha\beta}k^\alpha k^\beta = 0 \,, && \text{(E.1.12)} \\ \Gamma^0{}_{\alpha 0} &= \Gamma^\gamma{}_{\alpha\beta}k^\beta k_\gamma = 0 \,. && \text{(E.1.13)} \end{aligned}$$

The determinant of the full metric is independent of ϵ

$$|g_{\alpha\beta}| = |\eta_{\alpha\beta} - 2\epsilon H k_\alpha k_\beta| = |\eta_{\alpha\beta}| = const. \qquad \longrightarrow \qquad \Gamma^{\beta}{}_{\alpha\beta} = 0\,,$$

and the contracted Riemann tensor therefore reduces to

$$R_{\alpha\beta} = R^{\gamma}{}_{\alpha\gamma\beta} = \Gamma^{\gamma}{}_{\alpha\beta,\gamma} - \Gamma^{\gamma}{}_{\alpha\delta}\Gamma^{\delta}{}_{\beta\gamma}\,. \tag{E.1.14}$$

The simplest component is

$$R_{\alpha\beta}k^\alpha k^\beta = \Gamma^{\gamma}{}_{\alpha\beta,\gamma}k^\alpha k^\beta - \Gamma^{\gamma}{}_{\delta 0}\Gamma^{\delta}{}_{\gamma 0} = \Gamma^{\gamma}{}_{00,\gamma} - 2\Gamma^{\gamma}{}_{\alpha 0}k^{\alpha}{}_{,\gamma} \tag{E.1.15}$$
$$= 2\epsilon H||\dot{\mathbf{k}}||^2\,. \tag{E.1.16}$$

If the L.H.S. is zero then $||\dot{\mathbf{k}}|| = 0$ and so $\dot{\mathbf{k}}$ is a null-vector orthogonal to another null-vector, $\mathbf{k}$. Hence $\dot{\mathbf{k}}$ must be parallel to $\mathbf{k}$ and therefore $\mathbf{k}$ is a geodesic vector.

The Ricci tensor expanded as series in ϵ is given by

$$R_{\alpha\beta} = \epsilon \underset{1}{R}{}_{\alpha\beta} + \epsilon^2 \underset{2}{R}{}_{\alpha\beta} + \epsilon^3 \underset{3}{R}{}_{\alpha\beta} + \epsilon^4 \underset{4}{R}{}_{\alpha\beta}\,. \tag{E.1.17}$$

The vacuum Einstein's equations $R_{\alpha\beta} = 0$ imply that contributions of all orders must vanish. Let us evaluate all such components.

The highest components of the expansion for the Ricci tensor are

$$\underset{4}{R}{}_{\alpha\beta} = -\underset{2}{\Gamma}{}^{\rho}{}_{\alpha\sigma}\underset{2}{\Gamma}{}^{\sigma}{}_{\beta\rho} = 0\,, \tag{E.1.18}$$
$$\underset{3}{R}{}_{\alpha\beta} = -\underset{1}{\Gamma}{}^{\rho}{}_{\alpha\sigma}\underset{2}{\Gamma}{}^{\sigma}{}_{\beta\rho} - \underset{2}{\Gamma}{}^{\rho}{}_{\alpha\sigma}\underset{1}{\Gamma}{}^{\sigma}{}_{\beta\rho} = 4H^3||\dot{\mathbf{k}}||^2 k_\alpha k_\beta\,. \tag{E.1.19}$$

The next component of $R_{\alpha\beta}$ is

$$\underset{2}{R}{}_{\alpha\beta} = \underset{2}{\Gamma}{}^{\rho}{}_{\alpha\beta,\rho} - \underset{1}{\Gamma}{}^{\rho}{}_{\alpha\sigma}\underset{1}{\Gamma}{}^{\sigma}{}_{\beta\rho} \tag{E.1.20}$$
$$= 2H\left[(Hk_\alpha k_\beta)^{\cdot\cdot} + k^{\sigma}{}_{,\sigma}(Hk_\alpha k_\beta)^{\cdot} - H\dot{k}_\alpha \dot{k}_\beta\right] \tag{E.1.21}$$
$$-H^2\Phi k_\alpha k_\beta - 2Hk_{(\alpha}\psi_{\beta)}\,, \tag{E.1.22}$$
$$\tag{E.1.23}$$

where

$$\Phi = 4\eta^{\gamma\lambda}\eta^{\delta\mu}k_{[\lambda,\delta]}k_{[\mu,\gamma]}\,, \qquad \psi_\alpha = 2\dot{k}^\gamma (Hk_\alpha)_{,\gamma}\,. \tag{E.1.24}$$

Finally, the first component of the Ricci tensor expansion is

$$\underset{1}{R}{}_{\alpha\beta} = \underset{1}{\Gamma}{}^{\gamma}{}_{\alpha\beta,\gamma} \tag{E.1.25}$$
$$= Ak_\alpha k_\beta + 2k_{(\alpha}B_{\beta)} + X_{\alpha\beta}\,, \tag{E.1.26}$$

where

$$
\begin{aligned}
A &= \eta^{\lambda\gamma} H_{,\lambda\gamma} \,, \\
B_\beta &= -(Hk^\gamma)_{,\gamma\beta} + \frac{1}{H}\eta^{\lambda\gamma}(H^2 k_{\beta,\gamma})_{,\lambda} \,, \\
X_{\alpha\beta} &= -2H\left[(k_{(\alpha,\beta)}k^\gamma)_{,\gamma} + k_{(\alpha,|\gamma|}k^\gamma{}_{,\beta)} - \eta^{\lambda\gamma}k_{\alpha,\gamma}k_{\beta,\lambda}\right] \\
&\quad -2k^\gamma\left[H_{,(\alpha}k_{\beta),\gamma} + H_{,\gamma}k_{(\alpha,\beta)}\right] \\
&= -2H\left[\dot{k}_{(\alpha,\beta)} + k^\gamma{}_{,\gamma}k_{(\alpha,\beta)} - \eta^{\lambda\gamma}k_{\alpha,\gamma}k_{\beta,\lambda}\right] \\
&\quad -2\dot{H}k_{(\alpha,\beta)} - 2H_{,(\alpha}\dot{k}_{\beta)} \,.
\end{aligned} \qquad \text{(E.1.27)}
$$

Therefore, the Ricci tensor turns out to consist of three different contributions. In paper ([158]) we have shown that third order equations all imply that $\mathbf{k}$ must be geodesic; it must be also shearfree as a consequence of first order equations, whereas the solution for H comes from second order equations too.

The same treatment can be generalized to include the electromagnetic field, i.e. to the case of Kerr-Newman. In fact, even in the charged Kerr solution the congruence of $\mathbf{k}$-lines depend only on the rotation parameter a and not on the mass $\mathcal{M}$ or charge Q. Furthermore, the electromagnetic field is linear in Q and the metric is linear in $\mathcal{M}$ and Q^2, since the function H is obtained simply by replacing $\mathcal{M} \to \mathcal{M} - Q^2/(2r)$.

A wider class of solutions could be obtained by modifying the original "linear" Kerr-Schild ansatz by a "quadratic" one. Suppose that

$$
g_{\alpha\beta} = \eta_{\alpha\beta} - 2\lambda k_{(\alpha}p_{\beta)} + \mu k_\alpha k_\beta \,, \qquad ||\mathbf{k}|| = 0 \,, \quad \mathbf{k}\cdot\mathbf{p} = 0 \,, \quad \mathbf{p}\cdot\mathbf{p} = 1 \,,
$$

where the vector $\mathbf{p}$ has to be spacelike. Its inverse is similar,

$$
g^{\alpha\beta} = \eta^{\alpha\beta} + 2\lambda k^{(\alpha}p^{\beta)} + \mu' k^\alpha k^\beta \,, \qquad \mu + \mu' = \lambda^2 \,.
$$

The determinant of this metric is independent of λ and μ, and so the components of the Einstein/Riemann tensors are also polynomials in these parameters. It is possible that this ansatz can include a black hole interior. Presumably the null vector $\mathbf{k}$ would have to be the same as for Kerr(-Newman). This investigation is left for future works.

E.2. Rational metrics

We have considered a general metric of the form

$$ds^2 = \frac{\Delta}{H}\left[\frac{dx^2}{V} + \frac{dy^2}{W}\right] + \frac{d\Sigma^2(dx^3, dx^4)}{\Delta}, \tag{E.2.1}$$

where

$$d\Sigma^2 = -A(dx^3)^2 + 2C dx^3 dx^4 + B(dx^4)^2\,. \tag{E.2.2}$$

The field equations use the following quantities

$$\begin{aligned}
g_{ab,x}g^{ab}{}_{,x} &= \frac{Q_{ab,x}\tilde{Q}^{ab}{}_{,x}}{\Delta^2 VW} - \frac{\Delta_{,x}^2}{\Delta^2} - \left(\frac{\Delta_{,x}}{\Delta} + \frac{V_{,x}}{V}\right)^2, \\
g_{ab,y}g^{ab}{}_{,y} &= \frac{Q_{ab,y}\tilde{Q}^{ab}{}_{,y}}{\Delta^2 VW} - \frac{\Delta_{,y}^2}{\Delta^2} - \left(\frac{\Delta_{,y}}{\Delta} + \frac{W_{,y}}{W}\right)^2, \\
g_{ab,x}g^{ab}{}_{,y} &= \frac{Q_{ab,x}\tilde{Q}^{ab}{}_{,y}}{\Delta^2 VW} - \frac{\Delta_{,y}\Delta_{,x}}{\Delta^2} - \left(\frac{\Delta_{,x}}{\Delta} + \frac{V_{,x}}{V}\right)\left(\frac{\Delta_{,y}}{\Delta} + \frac{W_{,y}}{W}\right)
\end{aligned} \tag{E.2.3}$$

The functions A, B, C instead have to satisfy second order differential equations:

$$\left(\frac{AC_{,x} - CA_{,x}}{W\Delta^2}\right)_{,x} + \left(\frac{AC_{,y} - CA_{,y}}{V\Delta^2}\right)_{,y} = 0 \tag{E.2.4}$$

with the integrability conditions (or equivalently the equations for the derivatives of H)

$$\frac{A_{,x}B_{,y} + B_{,x}A_{,y} + 2C_{,x}C_{,y}}{VW\Delta^2} - \frac{2\Delta_{,x}\Delta_{,y}}{\Delta^2} - \frac{\dot{V}}{V}\frac{\dot{W}}{W} - \frac{\dot{V}}{V}\frac{H_{,y}}{H} - \frac{\dot{W}}{W}\frac{H_{,x}}{H} = 0,$$

$$\frac{A_{,x}B_{,x} + C_{,x}^2}{W\Delta^2} - \frac{A_{,y}B_{,y} + C_{,y}^2}{V\Delta^2} - \frac{V\Delta_{,x}^2 - W\Delta_{,y}^2}{\Delta^2} + \ddot{W} - \ddot{V} - \frac{\dot{V}H_{,x}}{H} + \frac{\dot{W}H_{,y}}{H} = 0\,.$$

This study is still preliminar (see also the section on "The Kerr-Newman solution," included in this report) but it seems promising in view of obtaining new exact solutions of Einstein's field equations for the case of stationary and axisymmetric spacetimes.

Bibliography

[1] Jantzen R.T., Carini P., Bini D., *Annals of Physics* **215**, (1992), 1–50.

[2] R. Ruffini and C. Sigismondi Eds., Nonlinear gravitodynamics. The Lense-Thirring effect. A documentary introduction to current research," World Scientific, Singapore (2003).

[3] D. Bini, P. Carini, R.T. Jantzen, *Int. J. Mod. Phys. D* **6**, (1997), 1–38.

[4] D. Bini, P. Carini, R.T. Jantzen, *Int. J. Mod. Phys. D* **6**, (1997), 143–198.

[5] Bini D., de Felice F. and Geralico A., *Class. Quantum Grav.* **21**, (2004), 5441.

[6] Bini D., de Felice F., Geralico A., Jantzen R. T., *Class. Quantum Grav.* **22**, (2005), 2947.

[7] Bini D., de Felice F., Geralico A., Jantzen R. T., *Class. Quantum Grav.* **23**, (2006), 3287.

[8] Teukolsky S.A., *Astrophys. J.* , **185**, (1973), 635 .

[9] Bini D., Cherubini C., Jantzen R.T. and Ruffini R., *Progr. Theor. Phys.* **107**, (2002), 967.

[10] Bini D., Cherubini C., Jantzen R.T. and Ruffini R. *Int. J. Mod. Phys. D* **12**, (2003), 1363.

[11] Bini D., Cherubini C, Jantzen R.T., *Class. Quantum Grav.* **19**, (2002), 5265-5272.

[12] V.B. Braginsky, C.M. Caves, and K.S. Thorne, *Phys. Rev.* **15** (1977), 2047.

[13] V.B. Braginsky, A.G. Polnarev, and K.S. Thorne, *Phys. Rev. Lett.* **53** (1984), 863.

[14] K.S. Thorne, R.H. Price, D.A. Macdonald, editors, "Black Holes: The Membrane Paradigm," Yale University Press, New Haven, 1986.

[15] K.S. Thorne, in "Quantum Optics, Experimental Gravitation, and Measurement Theory," (P. Meystre and M.O. Scully, Eds.), Plenum Press, New York and London, 1981, and in "Near Zero: New Frontiers

of Physics," (J.D. Fairbank, B.S. Deaver, Jr., C.W. Everitt, and P.F. Michelson, Eds.), Freeman, New York, 1988.

[16] R.L. Forward, *Proceedings of the IRE* **49** (1961), 892.

[17] J. Ehlers, *Akad. Wiss. Mainz Abh., Math.-Nat. Kl.*, Nr. 11 (1961), 793.

[18] S.W. Hawking and G.F.R. Ellis, "The Large Scale Structure of Space-Time," Cambridge University Press, Cambridge, 1973.

[19] D. Bini and R.T. Jantzen, *Proceedings of the XXIII Spanish Relativity Meeting on Reference Frames and Gravitomagnetism* (http://hades.eis.uva.es/EREs2000), (J.-F. Pascual-Snchez, L. Flora, A. San Miguel, and F. Vicente, Eds.), World Scientific, Singapore, 2001.

[20] L.D. Landau and E.M. Lifshitz, "The Classical Theory of Fields," Permagon Press, New York, 1975.

[21] C. Møller, "The Theory of Relativity," Oxford University Press, Oxford, First Edition: 1952, Second Edition: 1972.

[22] A.L. Zel'manov, *Dokl. Akad. Nauk. USSR* **107** (1956), 805 [Eng. trans. in *Sov. Phys. Doklady* **1** (1956), 227].

[23] A.L. Zel'manov, *Sov. Phys. Dokl.* **18** (1973), 231.

[24] C. Cattaneo, *Nuovo Cim.* **10** (1958), 318; **11** (1958), 733.

[25] C. Cattaneo, *Ann. Mat. Pura ed Appl.* **48** (1959), 361.

[26] C. Cattaneo, "Calcolo Differenziale Assoluto su una Varietá Riemanniana," Libreria E.V. Veschi, Rome, (1961).

[27] G. Ferrarese, *Rend. di Mat.* **22** (1963), 147; *Rend. di Mat.* **24** (1965), 57.

[28] E. Massa, *Gen. Relativ. Grav.* **5** (1974), 555; **5** (1974), 573; **5** (1974), 715.

[29] G.F.R. Ellis, in "General Relativity and Cosmology: Proceedings of Course 47 of the International School of Physics 'Enrico Fermi'," (R. Sachs, Ed.) Academic Press, New York, 1971.

[30] G.F.R. Ellis, in "Càrgese Lectures in Physics," Vol. 6, (E. Schatzman, Ed.), Gordon and Breach, New York, 1973.

[31] A. Lichnerowicz, *J. Math. Pures Appl.* **23** (1944), 37.

[32] Y. Choquet-Bruhat, *J. Rat. Mech. Anal.* **5** (1956), 951.

[33] R. Arnowit, S. Deser and C.W. Misner, in “Gravitation: An Introduction to Current Research,” (L. Witten, Ed.), Wiley, New York, 1962.

[34] C.W. Misner, K.S. Thorne and J.A. Wheeler, “Gravitation,” Freeman, San Francisco, 1973.

[35] M.A. Abramowicz, B. Carter, and J.P. Lasota, *Gen. Relativ. Grav.* **20** (1988), 1173.

[36] M.A. Abramowicz, *Mon. Not. R. Ast. Soc.* **245** (1990), 733.

[37] D. Bini, P. Carini, R.T. Jantzen, Wilkins D., *Phys. Rev. D* **49**, (1994), 2820–2827.

[38] D. Bini, P. Carini, R.T. Jantzen, *Class. Quantum Grav.* **12**, (1995), 2549–2563.

[39] D. Bini, R.T. Jantzen, A. Merloni, *Il Nuovo Cimento B*, **113**, (1998), 611–620.

[40] D. Bini, R.T. Jantzen, A. Merloni, *Class. Quantum Grav.* **16**, (1999), 1333–1348.

[41] D. Bini, F. de Felice , R.T. Jantzen, *Class. Quantum Grav.* **16**, (1999), 2105–2124.

[42] D. Bini, R.T. Jantzen, G. Miniutti, *Class. Quantum Grav.* **18**, (2001), 1–13.

[43] D. Bini, C. Cherubini, R.T. Jantzen, G. Miniutti, *Class. Quantum Grav.* **21**, (2004), 1987–1998.

[44] Bini D., Ruffini R., Spoliti G., *Il Nuovo Cimento B*, **112**, (1997), 173-180.

[45] Merloni A., Stella L., Vietri M., Bini D., *MNRAS*, **304**, (1999), 155-159.

[46] Bini D., Gemelli G., Ruffini R., *Phys. Rev. D*, **61**, (2000), 064013.

[47] Bini D., Jantzen R.T., *Class. Quantum Grav.* **17**, (2000), 1-11.

[48] Bini D., de Felice F., *Class. Quantum Grav.* **17**, (2000), 4627-4635.

[49] Bini D., Germani C., Jantzen R.T., *Int. J. Mod. Phys. D* **10**, (2001), 633-647.

[50] Bini D., Jantzen R.T., Mashhoon B., *Class. Quantum Grav.* **18**, (2001), 653-670.

[51] Sorge F., Bini D., de Felice F., *Class. Quantum Grav.* **18**, (2001), 2945-2958.

[52] Gemelli G., Bini D., de Felice F., *Int. J. Mod. Phys. D* **11**, (2002), 223.

[53] Bini D., Jantzen R.T., Mashhoon B., *Class. Quantum Grav.* **19**, (2002), 17-37.

[54] Cherubini C., Bini D., Capozziello S., Ruffini R., *Int. J. Mod. Phys. D* **11**, (2002), 827-841.

[55] Bini D., Jantzen R.T., Miniutti G., *Int. J. Mod. Phys. D* **11**, (2002), 1439-1450.

[56] Bini D., Cherubini C., Jantzen R.T., *Class. Quantum Grav.* **19**, (2002), 5481-5488.

[57] Bini D., Cherubini C, Jantzen R.T., Mashhoon B., *Class. Quantum Grav.* **20**, (2003), 457-468.

[58] Bini D, Cruciani G., Lunari A., *Class. Quantum Grav.* **20**, (2003), 341-350.

[59] Bini D., Cherubini C., Jantzen R.T., Mashhoon B., *Phys. Rev. D*, **67**, (2003), 084013.

[60] Bini D., Jantzen R.T., *Special Observers in the Kerr spacetime*, In "Advances in General Relativity and Cosmology", Ed. G. Ferrarese, Pitagora, pag. 287-296, 2003.

[61] Bini D., Jantzen R.T., *Gravitomagnetic clock effects in black hole spacetime*, In "General Relativity, Cosmology and Gravitational Lensing", Napoli Series on Physics and Astrophysics, pp. 17-28 Eds. G. Marmo, C. Rubano, P. Scudellaro, Bibliopolis, 2002.

[62] Bini D., Crosta M. T., de Felice F., *Class. Quantum Grav.* **20**, (2003), 4695-4706.

[63] Bini D., de Felice F., *Class. Quantum Grav.* **20**, (2003), 2251-2259.

[64] Bini D., Cherubini C., De Mattia M., Jantzen R.T., *Gen. Rel. Grav.* **35**, (2003), 2249-2260.

[65] Bini D., Cherubini C., Cruciani G., Lunari A., *Int. J. Mod. Phys. D* **12**, (2003), 1983-2000.

[66] Bini D., *Il Nuovo Cimento B*, **118**, (2003), 1055-1061.

[67] Bini D., Jantzen R.T., *Inertial forces: the special relativistic assessment*, in "Relativity in Rotating Frames (Relativistic physics in rotating reference frames) Ed. by G. Rizzi and M.L. Ruggiero Kluwer Academic Press, Fundamental Theories of Physics, vol. 135, 221-239, 2004.

[68] Bini D, Geralico A., *Int. J. Mod. Phys. D* **13**, (2004), 983-1003.

[69] Bini D., Cherubini C., *Class. Quantum Grav.* **21**, (2004), 4523-4529.

[70] Bini D., Cherubini C., Mashhoon B., *Phys. Rev. D*, **70**, (2004), 044020-7.

[71] Bini D., Cherubini C., Mashhoon B., *Class. Quantum Grav.* **21**, (2004), 3893-3908.

[72] Cherubini C., Bini D., Bruni M., Perjes Z., *Class. Quantum Grav.* **21**, (2004), 4833-4843.

[73] Bini D., Cherubini C., Cruciani G., Jantzen R.T., *Int. J. Mod. Phys. D* **13**, (2004), 1771-1803.

[74] Bini D., Jantzen R.T., *Il Nuovo Cimento B* **119**, (2004), 863-873.

[75] Bini D., *Il Nuovo Cimento B* **120**, (2005), 1021-1025..

[76] Cherubini C., Bini D., Bruni M., Perjes Z., *Class. Quantum Grav.* **22**, (2005), 1763-1768.

[77] Bini D., Lusanna L., Mashhoon B., *Int. J. Mod. Phys. D* **14**, (2005), 1413-1429.

[78] Bini D., De Paolis F., Geralico A., Ingrosso G., Nucita A. *Gen. Rel. Grav.* **37**, (2005), 1263-1276.

[79] Bini D., Geralico A., Jantzen R. T., *Class. Quantum Grav.* **22**, (2005), 4729-4742.

[80] Bini D., Cherubini C., Filippi S., Geralico A., *Class. Quantum Grav.* **22**, (2005), 5157-5168.

[81] Bini D., Geralico A., Jantzen R.T. *Class. Quantum Grav.* **23**, (2006), 3963-3981.

[82] Bini D., de Felice F., Geralico A. *Class. Quantum Grav.* **23**, (2006), 7603-7626.

[83] Bini D., de Felice F., Geralico A. *Phys. Rev. D*, **76**, (2007), 047502.

[84] Papapetrou A., *Proc. Roy. Soc. London* **209**, (1951), 248.

[85] Mathisson M., *Acta Phys. Polonica* **6**, (1937), 167 .

[86] Corinaldesi E. and Papapetrou A., *Proc. Roy. Soc. London* **209**, (1951), 259.

[87] Pirani F., *Acta Phys. Polon.* **15**, (1956), 389.

[88] Tulczyjew W., *Acta Phys. Polon.* **18**, (1959), 393.

[89] Dixon W. G., *Il Nuovo Cimento* **34**, (1964), 318.

[90] Dixon W. G., *Proc. Roy. Soc. London A* **314**, (1970), 499.

[91] Dixon W. G., *Proc. Roy. Soc. London A* **319**, (1970), 509.

[92] Dixon W. G., *Gen. Rel. Grav.* **4**, (1973), 199.

[93] Dixon W. G., *Philos. Trans. R. Soc. London. Ser. A* **277**, (1974), 59.

[94] Taub A. H., *J. Math. Phys.* **5**, (1964), 112.

[95] Mashhoon B., *J. Math. Phys.* **12**, (1971), 1075.

[96] Mashhoon B., *Ann. of Phys.* **89**, (1975), 254.

[97] Ehlers J. and Rudolph E., *Gen. Rel. Grav.* **8**, (1977), 197.

[98] Tod K. P., de Felice F. and Calvani M., *Il Nuovo Cimento B* **34**, (1976), 365.

[99] Souriau J. M., *C.R. Acad. Sc. Paris* **271**, (1970), 751.

[100] Souriau J. M., *C.R. Acad. Sc. Paris* **271**, (1970), 1086.

[101] Souriau J. M., *C.R. Acad. Sc. Paris* **274**, (1972), 1082.

[102] Souriau J. M., *Ann. Inst. H. Poincaré* **20**, (1974), 22.

[103] Bargman V., Michel L. and Telegdi V., *Phys. Rev. Lett.* **2**, (1959), 435.

[104] Bini D., de Felice F. and Geralico A., *Class. Quantum Grav.* **21**, (2004), 5427.

[105] Bini D., de Felice F., Geralico A., *Int. Jour. Mod. Phys. D* **14**, (2005), 1793.

[106] Bini D., de Felice F., Geralico A., Lunari A., *J. Phys. A: Math. Gen.* **38**, (2005), 1163.

[107] Bini D., Cherubini C., Geralico A. and Mashhoon B., *Class. Quantum Grav.* **22**, (2005), 709.

[108] Mashhoon B., Granwald F., Lichtnegger H. I. M., *Lectures Notes in Physics* **562**, (2001), 83.

[109] Cohen J. M. and Mashhoon B., *Phys. Lett. A* **181**, (1993), 353.

[110] Ciufolini I. and Ricci F., *Class. Quant. Grav.* **19**, (2002), 3863.

[111] Ciufolini I. and Ricci F., *Class. Quant. Grav.* **19**, (2002), 3875.

[112] Bini D., Cherubini C., Geralico A., Jantzen R. T., *Int. Jour. Mod. Phys. D* **15**, (2006), 737.

[113] Bini D., Fortini P., Geralico A., Ortolan A., *Class. and Quantum Grav.* **25**, (2008), 035005.

[114] Bini D., Fortini P., Geralico A., Ortolan A., *Class. and Quantum Grav.* **25**, (2008), 125007.

[115] Newman E. Penrose R. *J. Math. Phys.* **3**, (1962), 566.

[116] Geroch R., Held A. and Penrose R. *J. Math. Phys.* **14** (1973), 874.

[117] Bose K. S. *J. Math. Phys.* **16**, (1975), 772.

[118] Kinnersley W. *J. Math. Phys.* **10**, (1969), 1195.

[119] Regge T. Wheeler J. A *Phys. Rev.* **108**, (1957), 1063.

[120] Zerilli F. J. *Phys. Rev. D* **9**, (1974), 860.

[121] Ruffini R. in Physics and Astrophysics of Neutron Stars and Black Holes North-Holland, Amsterdam edited by Giacconi R. and Ruffini R. (1978).

[122] Chrzanowsky P. L., *Phys. Rev. D* **13**, (1976), 806.

[123] Lee C. H., Report IC/77/121 of the International Center for Theoretical Physics, Miramare, Trieste , (1977).

[124] Lee C. H., *J. Math. Phys* **17**, (1976), 1226.

[125] Press W. H. Teukolsky S., *Astrophys. J.* **185**, (1973), 649.

[126] Chandrasekhar S., The Mathematical Theory of Black Holes *Clarendon Press, Oxford*, (1983).

[127] Fackerell E. D., Proceedings of the Second Marcel Grossman Meeting on General Relativity (1982), Ed. Ruffini R., North-Holland, Amsterdam.

[128] Bellezza V. Ferrari V., *J. Math. Phys.* **25**, (1984), 1985.

[129] Cherubini C. and Ruffini R., *Il Nuovo Cimento B* **115**, (2000), 699.

[130] Bini D., Cherubini C. and Jantzen R.T., *Class. Quant. Grav.* **19**, (2002), 5265.

[131] D. Kramer, H. Stephani, E. Hertl, and M. McCallum, Exact solutions of Einstein's field equations *Cambridge University Press, Cambridge*, (1979).

[132] E. Fermi, *Il Nuovo Cimento* **22**, (1921), 176.

[133] E. T. Whittaker, *Proc. Roy. Soc.* **A116**, (1927), 720.

[134] E. T. Copson, *Proc. Roy. Soc.* **A118**, (1928), 184.

[135] R. Hanni, Junior Paper submitted to the Physics Department of Princeton University, (1970), unpublished.

[136] J. Cohen and R. Wald, *J. Math. Phys.* **12**, (1971), 1845.

[137] R. Hanni and R. Ruffini, *Phys. Rev. D* **8**, (1973), 3259.

[138] J. Bičák and L. Dvořák, *Gen. Relativ. Grav.* **7**, (1976), 959.

[139] B. Linet, *J. Phys. A: Math. Gen.* **9**, (1976), 1081.

[140] B. Leaute and B. Linet, *Phys. Lett.* **58A**, (1976), 5.

[141] D. Bini, A. Geralico, and R. Ruffini, *Phys. Lett.* **A360**, (2007), 515.

[142] D. Bini, A. Geralico, and R. Ruffini, *Phys. Rev. D* **75**, (2007), 044012.

[143] D. Bini, A. Geralico, and R. Ruffini, *Phys. Rev. D* **77**, (2008), 064020.

[144] M. Johnston, R. Ruffini, and F. J. Zerilli, *Phys. Lett.* **B49**, (1974), 185.

[145] M. Johnston, R. Ruffini, and F. J. Zerilli, *Phys. Rev. Lett.* **31**, (1973), 1317.

[146] W. B. Bonnor, *Class. Quant. Grav.* **10**, (1993), 2077.

[147] S. M. Majumdar, *Phys. Rev.* **72**, (1947), 390.

[148] A. Papapetrou, Proc. R. Irish Acad. **51**, (1947), 191.

[149] G. A. Alekseev and V. A. Belinski, *Phys. Rev. D* **76**, (2007), 021501(R).

[150] G. A. Alekseev, *Sov. Phys. Dokl.* **30**, (1985), 565; *Proc. Steklov Inst. Math.* **3**, (1988), 215.

[151] D. Bini, C. Germani, and R. T. Jantzen, *Int. J. Mod. Phys. D* **10**, (2001), 633.

[152] R. P. Kerr and A. Schild, "A new class of vacuum solutions of the Einstein field equations," in *Atti del convegno sulla relatività generale; problemi dell'energia e onde gravitationali*, G. Barbera, Ed., p. 173 (Firenze, 1965).

[153] R. P. Kerr and A. Schild, Proc. Symp. Appl. Math. **17**, 199 (1965).

[154] R. P. Kerr, Phys. Rev. Lett. **11**, 237 (1963).

[155] Goldberg J.N. and Sachs R.K., Acta Phys. Polon. **22** Suppl., 13 (1962).

[156] Bini D., Jantzen R. T., Stella L., Classical and Quantum Gravity, **26**, 055009 (2009).

[157] De Felice F., Bini D., *Classical Measurements in Curved Space-Times* Series: Cambridge Monographs on Mathematical Physics, (Cambridge Univ. Press, Cambridge, 2010).

[158] Bini D., Geralico A., Kerr R.P., International Journal of Geometric Methods in Modern Physics (IJGMMP) **7**, 693-703 (2010).

[159] Bini D., Geralico A., Classical and Quantum Gravity, **27**, 185014 (2010).

[160] Bini D. , Geralico A., Jantzen R.T., General Relativity and Gravitation, vol. 43, 18371853, 2011.

[161] Bini D., Geralico A., Jantzen R. T., General Relativity and Gravitation, vol. 43, 959-975, 2011.

[162] Bini D., Geralico A., Jantzen R. T., Semeřák O. and Stella L., Classical and Quantum Gravity, vol. 28 035008 (21pp), 2011.

[163] Bini D. , de Felice F., Geralico A., Classical and Quantum Gravity, vol. 28 225012, 2011.

[164] Bini D., Esposito G., Geralico A., General Relativity and Gravitation, to appear, 2011.

[165] Bini D., Geralico A., Jantzen R. T. and Semeřák O. Classical and Quantum Gravity, to appear, 2011.

[166] Bini D. and Geralico A. Physical Review D, to appear, 2011.

[167] Bini D., Geralico A., Jantzen R. T. General Relativity and Gravitation, to appear, 2011.

[168] Bini D., Fortini P., Haney M., Ortolan A. Classical and Quantum Gravity, vol. 28, 235007, 2011.

[169] Wen-Biao Han and Zhoujian Cao, Physical Review D, vol. 84, 044014, 2011.